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4D, $\mathcal{N} = 1$ Supergravity Genomics

Isaac Chappell¹, S. James Gates, Jr.², William D. Linch III³, James Parker⁴,
 Stephen Randall⁵, Alexander Ridgway⁶, and Kory Stiffler⁷

*Center for String and Particle Theory
 Department of Physics, University of Maryland
 College Park, MD 20742-4111 USA*

ABSTRACT

The off-shell representation theory of 4D, $\mathcal{N} = 1$ supermultiplets can be categorized in terms of distinct irreducible graphical representations called adinkras. Recent evidence has emerged pointing to the existence of three such fundamental adinkras associated with distinct equivalence classes of a Coxeter group. A partial description of these adinkras is given in terms of two types, termed cis- and trans-adinkras (the latter being a degenerate doublet) in analogy to enantiomers in chemistry. Through a new and simple procedure that uses adinkras, we find the irreducible off-shell adinkra representations of 4D, $\mathcal{N} = 1$ supergravity, in the old-minimal, non-minimal, and conformal formulations. We categorize these representations in terms of their supersymmetry ‘enantiomer’ numbers: the number of cis- (n_c) and trans- (n_t) adinkras in the representation.

¹isaac.chappell@gmail.com

²gatess@wam.umd.edu

³wdlinch3@gmail.com

⁴jp@jamesparker.me

⁵stephenrandall@gmail.com

⁶alecridgway@gmail.com

⁷kstiffle@gmail.com

1 Introduction

The subject of off-shell 4D, $\mathcal{N} = 1$ supergravity continues to be one that reveals surprises even today. Among the earliest off-shell formulations of 4D, $\mathcal{N} = 1$ supergravity were those in Refs. [1, 2]. Shortly thereafter the topic received much more attention due to the near simultaneous work in Refs. [3, 4]. Recently, the auxiliary field structure of the various off-shell supergravity theories [5] has been exploited to yield new insights into supersymmetric field theories in rigid, curved backgrounds [6, 7].

A long time ago, 4D, $\mathcal{N} = 1$ Poincaré supergravity, in its minimal and non-minimal off-shell representations, was succinctly written in terms of a real parameter n that originated in a superspace prepotential analysis [8]. There the discussion was given using superspace in terms of non-Riemannian geometry, with torsion. We now join this old effort to our current one: that of describing the representation theory for all off-shell 4D, $\mathcal{N} = 1$ supersymmetric (SUSY) multiplets in a systematic form based upon a group theory-like foundation. This is one part of a larger quest to extend such an understanding to *all* off-shell supersymmetric systems.

In Ref. [8] supergravity was analyzed simultaneously in both the old-minimal and non-minimal formulations in terms of a single parameter n . This parameter takes the value $n = -\frac{1}{3}$ for the old-minimal formulation (mSG), $n = 0$ for the so-called new-minimal supergravity (ν SG) and any $n \neq -\frac{1}{3}, 0$ for the non-minimal representation (\bar{m} SG).⁸ We use the same parameter n in this paper. Besides the aforementioned use of these various formulations to describe rigid curved backgrounds, the non-minimal generalizations have certain roles to play in other areas. In the covariant formulation of the $\mathcal{N} = 1$ heterotic string in four dimensions, for example, the explicit form of the graviton vertex operator singles out the $n = -1$ non-minimal theory. This same theory was also shown in Ref. [11] to belong to a family of $\mathcal{N} = 1$ higher-spin theories for which it is possible to find a non-linearly realized $\mathcal{N} = 2$ extension.

In our quest to build off-shell SUSY representation theory, we use graphical representation tools – adinkras – as the irreducible representations for such SUSY systems. Adinkras are built on $\mathcal{GR}(d, N)$ algebras (the so-called *garden algebras*) which are the algebras of N general, real $d \times d$ matrices that describe N supersymmetries between d bosons and d fermions [12, 13]. In two previous Refs. [12, 13] we found the adinkraic representations for five well-known systems: the chiral, vector, tensor, real scalar,

⁸There is, in fact, yet another theory at the linearized level discovered in Ref. [9] and explained in Ref. [10]. This “new-new-minimal” theory ($\nu\nu$ SG) is minimal in that it is described by (12|12) degrees of freedom and is very similar in structure to the new-minimal formulation.

and complex linear supermultiplets. A succinct way of classifying these representations exists in terms of two SUSY *enantiomer* numbers n_c and n_t . As explained in Ref. [13], these numbers were named after their analogy with chiral enantiomers in chemistry. These numbers encode the number of fundamental cis- (n_c) and trans- (n_t) valise adinkras that are used to represent the system.

To make an analogy with $su(3)$, the value of the enantiomer numbers for adinkras now appear to be analogous to using the eigenvalues of hypercharge for anti-quarks. It is well known that the anti-up-quark and anti-down-quark possess the same hypercharge value and thus constitute a degenerate doublet in this respect. The anti-strange-quark is the only one with a distinct value. The tensor supermultiplet and vector supermultiplet possess the same enantiomer values, while the chiral supermultiplet is the one with a distinct value. We will show in this paper that this adinkraic degeneracy continues for larger SUSY representations.

Once one becomes aware of the $su(3)$ isospin operator and its action upon the various quarks, the anti-up-quark and anti-down-quark can be seen to be distinct. At the time Refs. [12, 13] were completed, all evidence pointed toward there being one cis- and one trans-adinkra representation. The chiral multiplet was shown to exist in the cis-adinkra representation, the vector and tensor multiplets in the trans-adinkra representation. These works found no adinkraic distinction between the tensor and vector multiplets. More recently Ref. [14] revealed the existence of a $\mathcal{GR}(d,N)$ operator (roughly analogous to an $su(3)$ isospin operator) that shows the vector and tensor supermultiplets in the space of adinkras may be regarded as distinct. This evidence is embedded in a definition of equivalence classes of the $\mathcal{GR}(d,N)$ algebras and is related to permutation elements. While we are aware of this distinction, we will not use this finer definition in the analysis of this paper. Work on the understanding of this additional structure was begun in Ref. [15].

1.1 The Adinkranization Process

In this section we give only an overview of the adinkranization process and specifically how it will relate to the supergravity multiplets investigated in this work. For a more complete review of the adinkranization process, we turn the reader to Refs. [12, 13]. A 4D, $\mathcal{N} = 1$ theory can be dimensionally reduced to a 1D, $N = 4$ theory by considering all the fields to be dependent on only one coordinate. We consider fields to have only time dependence and refer to the reduction as the *0-brane reduction*. Once a system has been reduced to the 0-brane, adinkra pictures can be drawn which faithfully

reproduce *all* the information of the 0-brane transformation laws

$$D_I \Phi_i = i(L_I)_i^{\hat{j}} \Psi_{\hat{j}} \quad , \quad D_I \Psi_{\hat{i}} = (R_I)_i^{\hat{j}} \dot{\Phi}_{\hat{j}} \quad (1.1)$$

between a collection of bosons Φ_i and fermions $\Psi_{\hat{i}}$ with $i = 1, 2, \dots, d_b$ and $\hat{j} = 1, 2, \dots, d_f$ counting the number of bosons and fermions, respectively. In the following, the notation $(d_b|d_f)$ refers to a system with d_b bosons and d_f fermions in valise form, where all the zero-brane fields have the same engineering dimension as in Eq. (1.1). For 1D, $\mathcal{N} = 4$ there are only two distinct irreducible $(4|4)$ transformation laws and thus two distinct adinkraic representations: the aforementioned cis- and trans-valise adinkras [12].

In Ref. [10], 4D, $\mathcal{N} = 1$ off-shell linearized supergravity was investigated in terms of superprojectors. There it was shown that all $(12|12)$ minimal models (mSG, ν SG, and new-new-minimal ($\nu\nu$ SG) supergravity) could be formulated in terms of two superprojectors. All models with three superprojectors led to reducible $(16|16)$ models, and models with four superprojectors were the non-minimal models, parameterized by a real number n .

Stated another way, the various supergravity representations can be thought of in terms of which compensator is added to the base $(8|8)$ 4D, $\mathcal{N} = 1$ conformal supergravity (cSG). For mSG, the compensator is a chiral superfield, σ . For ν SG or $\nu\nu$ SG, the compensator is a real linear superfield \mathcal{U} or U , respectively, where the difference is in the gauge transformation. For $\not\mu$ SG, the compensator is a complex linear superfield $\Sigma = a\sigma + b\mathcal{U} + cU$ with a , b , and c constants.

We can thus succinctly denote the representation sizes as $(4k|4k)$, with $k = 2$ cSG, $k = 3$ mSG, ν SG, or $\nu\nu$ SG, $k = 4$ reducible, and $k = 5$ $\not\mu$ SG. This leads to k adinkras for a representation: The question is how many cis- and trans-adinkras are there and which component fields are part of which irreducible representation. In Section 2 we reveal a simple procedure to uncover these irreducible submultiplets. Throughout Sections 2, 3, and 4 we will show how the adinkras for the compensator field and cSG emerge to compose the full representations. In Sec. 5, we explain the pattern from $SO(4)$ representation theory.

2 4D, $\mathcal{N} = 1$ Minimal SG ($n = -1/3$)

The linearized theory of 4D, $\mathcal{N} = 1$ Poincaré old-minimal supergravity (mSG) contains the real component fields of a scalar auxiliary field S , pseudoscalar auxiliary

field P , axial vector auxiliary field A_μ , Majorana gravitino $\psi_{\mu a}$, and graviton $h_{\mu\nu}$ [8, 16].

2.1 Transformation Laws

We use the transformation laws

$$D_a S = -\frac{1}{2}([\gamma^\mu, \gamma^\nu])_a{}^b \partial_\mu \psi_{\nu b} \quad (2.1a)$$

$$D_a P = \frac{1}{2}(\gamma^5 [\gamma^\mu, \gamma^\nu])_a{}^b \partial_\mu \psi_{\nu b} \quad (2.1b)$$

$$D_a A_\mu = i(\gamma^5 \gamma^\nu)_a{}^b \partial_{[\nu} \psi_{\mu]b} - \frac{1}{2} \epsilon_\mu{}^{\nu\alpha\beta} (\gamma_\nu)_a{}^b \partial_\alpha \psi_{\beta b} \quad (2.1c)$$

$$D_a h_{\mu\nu} = \frac{1}{2}(\gamma_{(\mu})_a{}^b \psi_{\nu)b} \quad (2.1d)$$

$$D_a \psi_{\mu b} = -\frac{i}{3}(\gamma_\mu)_{ab} S - \frac{1}{3}(\gamma^5 \gamma_\mu)_{ab} P + \frac{2}{3}(\gamma^5)_{ab} A_\mu + \frac{1}{6}(\gamma^5 [\gamma_\mu, \gamma^\nu])_{ab} A_\nu + \\ -\frac{i}{2}([\gamma^\alpha, \gamma^\beta])_{ab} \partial_\alpha h_{\beta\mu} \quad (2.1e)$$

which are a symmetry of the Lagrangian

$$\mathcal{L}_{mSG} = -\frac{1}{2} \partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} + \frac{1}{2} \partial^\alpha h \partial_\alpha h - \partial^\alpha h \partial^\beta h_{\alpha\beta} + \partial^\mu h_{\mu\nu} \partial_\alpha h^{\alpha\nu} \\ -\frac{1}{3} S^2 - \frac{1}{3} P^2 + \frac{1}{3} A_\mu A^\mu - \frac{1}{2} \psi_{\mu a} \epsilon^{\mu\nu\alpha\beta} (\gamma^5 \gamma_\nu)^{ab} \partial_\alpha \psi_{\beta b} \quad (2.2)$$

up to total derivatives

$$D_a \mathcal{L}_{mSG} = 0 + \text{total derivatives.} \quad (2.3)$$

In the above and hereafter, h is the trace of the graviton

$$h \equiv \eta^{\mu\nu} h_{\mu\nu} \quad . \quad (2.4)$$

A direct calculation reveals the following algebra

$$\{D_a, D_b\} S = 2i(\gamma^\mu)_{ab} \partial_\mu S \quad , \quad \{D_a, D_b\} P = 2i(\gamma^\mu)_{ab} \partial_\mu P \quad , \quad (2.5a)$$

$$\{D_a, D_b\} A_\nu = 2i(\gamma^\mu)_{ab} \partial_\mu A_\nu \quad , \quad (2.5b)$$

$$\{D_a, D_b\} h_{\mu\nu} = 2i(\gamma^\alpha)_{ab} \partial_\alpha h_{\mu\nu} - i(\gamma^\alpha)_{ab} \partial_{(\mu} h_{\nu)\alpha} \quad , \quad (2.5c)$$

$$\{D_a, D_b\} \psi_{\mu c} = 2i(\gamma^\alpha)_{ab} \partial_\alpha \psi_{\mu c} - i \partial_\mu \varphi_{abc} \quad . \quad (2.5d)$$

The extra term on the right hand side of Eq. (2.5d) is given by

$$\begin{aligned}\varphi_{abc} = & \frac{5}{4}(\gamma^\alpha)_{ab}\partial_\mu\psi_{\alpha c} + \frac{1}{8}\left((\gamma^5[\gamma^\beta, \gamma^\alpha])_{ab}(\gamma^5\gamma_\beta)_c{}^d + ([\gamma^\beta, \gamma^\alpha])_{ab}(\gamma_\beta)_c{}^d\right)\partial_\mu\psi_{\alpha d} \\ & + \frac{1}{8}(\gamma_\beta)_{ab}[\gamma^\beta, \gamma^\alpha]_c{}^d\partial_\mu\psi_{\alpha d} \quad .\end{aligned}\quad (2.6)$$

Comparing the RHS's of Eqs. (2.5) for the graviton ($h_{\mu\nu}$) and gravitino ($\psi_{\mu a}$) to those for the other fields, there are extra terms that are consequences of the well known gauge symmetries of the Lagrangian

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu \quad (2.7)$$

$$\psi_{\mu a} \rightarrow \psi_{\mu a} + \partial_\mu\epsilon_a \quad (2.8)$$

for arbitrary infinitesimal vectors Λ_μ and spinors ϵ_a .

2.2 One Dimensional Reduction

In the temporal gauge

$$h_{0\mu} = \psi_{0a} = 0 \quad (2.9)$$

the Lagrangian reduced to the 0-brane becomes

$$\begin{aligned}\mathcal{L}_{mSG}^{(0)} = & \dot{h}_{12}^2 + \dot{h}_{13}^2 + \dot{h}_{23}^2 - \dot{h}_{11}\dot{h}_{22} - \dot{h}_{11}\dot{h}_{33} - \dot{h}_{22}\dot{h}_{33} \\ & - \frac{1}{3}S^2 - \frac{1}{3}P^2 - \frac{1}{3}A_0^2 + \frac{1}{3}A_1^2 + \frac{1}{3}A_2^2 + \frac{1}{3}A_3^2 \\ & + i\left(-\psi_{31}\dot{\psi}_{12} + \psi_{32}\dot{\psi}_{11} - \psi_{33}\dot{\psi}_{14} + \psi_{34}\dot{\psi}_{13} - \psi_{11}\dot{\psi}_{23} + \psi_{12}\dot{\psi}_{24} \right. \\ & \left. + \psi_{13}\dot{\psi}_{21} - \psi_{14}\dot{\psi}_{22} - \psi_{21}\dot{\psi}_{34} - \psi_{22}\dot{\psi}_{33} + \psi_{23}\dot{\psi}_{32} + \psi_{24}\dot{\psi}_{31}\right) \quad .\end{aligned}\quad (2.10)$$

In the above and hereafter we use the shorthand for time derivatives

$$\dot{h}_{11} = \partial_0 h_{11} \quad . \quad (2.11)$$

The 0-brane reduced transformation laws are displayed in Tables 1 and 2. Table 3 shows the transformation laws for the cis- and trans-valise representations simultaneously via the parameter $\chi_0 = n_c - n_t$ where $\chi_0 = 1$ for cis-valise and $\chi_0 = -1$ for trans-valise. Fig. 1 shows the cis- and trans-valise adinkras where the colors encode the D-transformations between the fields with the dashed (solid) lines encoding an overall minus (plus) sign. Also, in translating adinkras to transformation laws there

is an overall factor of the imaginary number i multiplying the boson in transformations from fermion to boson. Our last rule in the translation is that there is a time derivative on the field in the lower node in moving from an upper node to a lower node.

Table 1: Supergravity transformation laws for bosons in temporal gauge, Eq.(2.9), and reduced to the 0-brane.

	D ₁	D ₂	D ₃	D ₄
S	$-\psi_{11} - \psi_{23} + \psi_{32}$	$\psi_{12} - \psi_{24} + \psi_{31}$	$\psi_{13} - \psi_{21} - \psi_{34}$	$-\psi_{14} - \psi_{22} - \psi_{33}$
A_2	$\frac{1}{2}\psi_{11} - \psi_{23} - \frac{1}{2}\psi_{32}$	$-\frac{1}{2}\psi_{12} - \psi_{24} - \frac{1}{2}\psi_{31}$	$\frac{1}{2}\psi_{13} + \psi_{21} - \frac{1}{2}\psi_{34}$	$-\frac{1}{2}\psi_{14} + \psi_{22} - \frac{1}{2}\psi_{33}$
h_{13}	$\frac{1}{2}\psi_{11} + \frac{1}{2}\psi_{32}$	$-\frac{1}{2}\psi_{12} + \frac{1}{2}\psi_{31}$	$\frac{1}{2}\psi_{13} + \frac{1}{2}\psi_{34}$	$-\frac{1}{2}\psi_{14} + \frac{1}{2}\psi_{33}$
h_{11}	ψ_{12}	ψ_{11}	ψ_{14}	ψ_{13}
h_{22}	$-\psi_{24}$	ψ_{23}	ψ_{22}	$-\psi_{21}$
h_{33}	ψ_{31}	$-\psi_{32}$	ψ_{33}	$-\psi_{34}$
A_0	$\psi_{13} - \psi_{21} - \psi_{34}$	$-\psi_{14} - \psi_{22} - \psi_{33}$	$\psi_{11} + \psi_{23} - \psi_{32}$	$-\psi_{12} + \psi_{24} - \psi_{31}$
A_1	$-\psi_{13} - \frac{1}{2}\psi_{21} - \frac{1}{2}\psi_{34}$	$-\psi_{14} + \frac{1}{2}\psi_{22} + \frac{1}{2}\psi_{33}$	$\psi_{11} - \frac{1}{2}\psi_{23} + \frac{1}{2}\psi_{32}$	$\psi_{12} + \frac{1}{2}\psi_{24} - \frac{1}{2}\psi_{31}$
h_{23}	$\frac{1}{2}\psi_{21} - \frac{1}{2}\psi_{34}$	$-\frac{1}{2}\psi_{22} + \frac{1}{2}\psi_{33}$	$\frac{1}{2}\psi_{23} + \frac{1}{2}\psi_{32}$	$-\frac{1}{2}\psi_{24} - \frac{1}{2}\psi_{31}$
P	$-\psi_{14} - \psi_{22} - \psi_{33}$	$-\psi_{13} + \psi_{21} + \psi_{34}$	$\psi_{12} - \psi_{24} + \psi_{31}$	$\psi_{11} + \psi_{23} - \psi_{32}$
A_3	$\frac{1}{2}\psi_{14} + \frac{1}{2}\psi_{22} - \psi_{33}$	$-\frac{1}{2}\psi_{13} + \frac{1}{2}\psi_{21} - \psi_{34}$	$-\frac{1}{2}\psi_{12} + \frac{1}{2}\psi_{24} + \psi_{31}$	$\frac{1}{2}\psi_{11} + \frac{1}{2}\psi_{23} + \psi_{32}$
h_{12}	$-\frac{1}{2}\psi_{14} + \frac{1}{2}\psi_{22}$	$\frac{1}{2}\psi_{13} + \frac{1}{2}\psi_{21}$	$\frac{1}{2}\psi_{12} + \frac{1}{2}\psi_{24}$	$-\frac{1}{2}\psi_{11} + \frac{1}{2}\psi_{23}$

Table 2: Supergravity transformation laws for fermions in temporal gauge, Eq.(2.9), and reduced to the 0-brane.

	D ₁	D ₂	D ₃	D ₄
ψ_{13}	$\frac{1}{3}iA_0 - \frac{2}{3}iA_1$	$-\frac{1}{3}iA_3 - \frac{1}{3}iP + ih_{12}$	$\frac{1}{3}iA_2 + \frac{1}{3}iS + ih_{13}$	ih_{11}
ψ_{21}	$-\frac{1}{3}iA_0 - \frac{1}{3}iA_1 + ih_{23}$	$\frac{1}{3}iA_3 + \frac{1}{3}iP + ih_{12}$	$\frac{2}{3}iA_2 - \frac{1}{3}iS$	$-ih_{22}$
ψ_{34}	$-\frac{1}{3}iA_0 - \frac{1}{3}iA_1 - ih_{23}$	$-\frac{2}{3}iA_3 + \frac{1}{3}iP$	$-\frac{1}{3}iA_2 - \frac{1}{3}iS + ih_{13}$	$-ih_{33}$
ψ_{14}	$\frac{1}{3}iA_3 - \frac{1}{3}iP - ih_{12}$	$-\frac{1}{3}iA_0 - \frac{2}{3}iA_1$	ih_{11}	$-\frac{1}{3}iA_2 - \frac{1}{3}iS - ih_{13}$
ψ_{22}	$\frac{1}{3}iA_3 - \frac{1}{3}iP + ih_{12}$	$-\frac{1}{3}iA_0 + \frac{1}{3}iA_1 - ih_{23}$	ih_{22}	$\frac{2}{3}iA_2 - \frac{1}{3}iS$
ψ_{33}	$-\frac{2}{3}iA_3 - \frac{1}{3}iP$	$-\frac{1}{3}iA_0 + \frac{1}{3}iA_1 + ih_{23}$	ih_{33}	$-\frac{1}{3}iA_2 - \frac{1}{3}iS + ih_{13}$
ψ_{11}	$\frac{1}{3}iA_2 - \frac{1}{3}iS + ih_{13}$	ih_{11}	$\frac{1}{3}iA_0 + \frac{2}{3}iA_1$	$\frac{1}{3}iA_3 + \frac{1}{3}iP - ih_{12}$
ψ_{23}	$-\frac{2}{3}iA_2 - \frac{1}{3}iS$	ih_{22}	$\frac{1}{3}iA_0 - \frac{1}{3}iA_1 + ih_{23}$	$\frac{1}{3}iA_3 + \frac{1}{3}iP + ih_{12}$
ψ_{32}	$-\frac{1}{3}iA_2 + \frac{1}{3}iS + ih_{13}$	$-ih_{33}$	$-\frac{1}{3}iA_0 + \frac{1}{3}iA_1 + ih_{23}$	$\frac{2}{3}iA_3 - \frac{1}{3}iP$
ψ_{12}	ih_{11}	$-i\frac{1}{3}A_2 + i\frac{1}{3}S - ih_{13}$	$-i\frac{1}{3}A_3 + i\frac{1}{3}P + ih_{12}$	$-i\frac{1}{3}A_0 + i\frac{2}{3}A_1$
ψ_{24}	$-ih_{22}$	$-\frac{2}{3}iA_2 - \frac{1}{3}iS$	$\frac{1}{3}iA_3 - \frac{1}{3}iP + ih_{12}$	$\frac{1}{3}iA_0 + \frac{1}{3}iA_1 - ih_{23}$
ψ_{31}	ih_{33}	$-\frac{1}{3}iA_2 + \frac{1}{3}iS + ih_{13}$	$\frac{2}{3}iA_3 + \frac{1}{3}iP$	$-\frac{1}{3}iA_0 - \frac{1}{3}iA_1 - ih_{23}$

Table 3: Transformation laws for an arbitrary *cis*- ($\chi_0 = 1$) or *trans*-valise ($\chi_0 = -1$) adinkra system. The Φ_i are bosons and the Ψ_i are fermions. These laws are encoded by the *cis*- and *trans*-valise adinkras in Fig. 1.

	D ₁	D ₂	D ₃	D ₄
Φ_1	$i\Psi_1$	$i\Psi_2$	$i\chi_0\Psi_3$	$-i\Psi_4$
Φ_2	$i\Psi_2$	$-i\Psi_1$	$i\chi_0\Psi_4$	$i\Psi_3$
Φ_3	$i\Psi_3$	$-i\Psi_4$	$-i\chi_0\Psi_1$	$-i\Psi_2$
Φ_4	$i\Psi_4$	$i\Psi_3$	$-i\chi_0\Psi_2$	$i\Psi_1$

	D ₁	D ₂	D ₃	D ₄
Ψ_1	Φ_1	$-\Phi_2$	$-\chi_0\Phi_3$	Φ_4
Ψ_2	Φ_2	Φ_1	$-\chi_0\Phi_4$	$-\Phi_3$
Ψ_3	Φ_3	Φ_4	$\chi_0\Phi_1$	Φ_2
Ψ_4	Φ_4	$-\Phi_3$	$\chi_0\Phi_2$	$-\Phi_1$

From Fig. 1, we can see that each D-transformation for an adinkraic representation is a bijective map between fermions and bosons. The transformations in Table 1 map twelve bosons into twelve linear combinations of fermions that are *different* for each D-transformation. Similarly, Table 2 maps twelve fermions into twelve linear combinations of bosons that are *different* for each D-transformation. For the (12|12) mSG system to be represented in terms of the irreducible (4|4) *cis*- and *trans*-valise

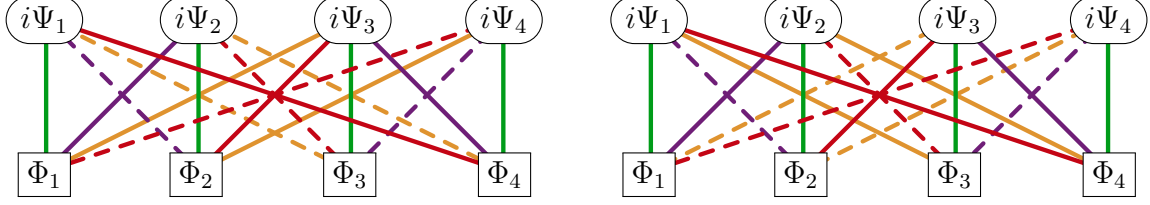


Figure 1: The *cis-* (left) and *trans-* (right) valise adinkras encoding the transformation laws in Table 3.

adinkras, there must be three distinct sets of linear combinations that define the nodes in the *cis-* or *trans-* valise adinkra and collapse Tables 1 and 2 to Table 3.

Tables 1 and 2 are organized horizontally into groups of three fields that can combine in linear combinations to form the nodes of an adinkra. Finding the linear combinations that collapse Tables 1 and 2 to either the $(4|4)$ *cis-* valise or the $(4|4)$ *trans-* valise basis in Table 3 and Fig. 1 will automatically give us the SUSY enantiomer numbers n_c and n_t for mSG. Our strategy is to find the number n_c of ways Tables 1 and 2 collapse to Table 3 under the *cis-* valise choice ($\chi_0 = 1$) and the number n_t of ways Tables 1 and 2 collapse to Table 3 under the *trans-* valise choice ($\chi_0 = -1$).

Starting with a seed linear combination:

$$i\Psi_1 = -u_1\dot{\psi}_{13} + u_2\dot{\psi}_{21} + u_3\dot{\psi}_{34} \quad (2.12)$$

where u_1 , u_2 , and u_3 are arbitrary constants, we can calculate the D-operator on the Ψ_1 node through both Tables 2 and 3 to define another node. We continue the process until we have defined all the nodes. We will see that after four iterations, each with a different color, choosing the *cis-* or *trans-* valise representation, $\chi_0 = 1$ or $\chi_0 = -1$ respectively, will lead to constraints on u_1 , u_2 , and u_3 . A simple analysis of these constraints will tell us:

1. The nodal field content of the *cis-* and *trans-* valise adinkras in the representation.
2. The number of *cis-* (n_c) and *trans-* valise (n_t) adinkras that compose the representation.

We proceed by calculating $\mathbf{D}_1\Psi_1$ from Table 3 and equating this to $\mathbf{D}_1\Psi_1$ calculated from Eq. (2.12) and Table 2. This defines Φ_1 :

$$\begin{aligned}
\textcolor{green}{D}_1 \Psi_1 &= \dot{\Phi}_1 \\
&= \frac{1}{3}(-u_1 - u_2 - u_3)\dot{A}_0 + \frac{1}{3}(2u_1 - u_2 - u_3)\dot{A}_1 + (u_2 - u_3)\ddot{h}_{23} \quad . \quad (2.13)
\end{aligned}$$

Next, we calculate $\textcolor{violet}{D}_2 \Phi_1$ through both Tables 1 and 3, defining Ψ_2 :

$$\begin{aligned}
\textcolor{violet}{D}_2 \Phi_1 &= i\Psi_2 \\
&= \frac{1}{3}(-u_1 + 2u_2 + 2u_3)\dot{\psi}_{14} + \frac{1}{3}(2u_1 - u_2 + 2u_3)\dot{\psi}_{22} \\
&\quad + \frac{1}{3}(2u_1 + 2u_2 - u_3)\dot{\psi}_{33} \quad . \quad (2.14)
\end{aligned}$$

Following this with a calculation of $\textcolor{brown}{D}_3 \Psi_2$ gives us Φ_4 :

$$\begin{aligned}
\textcolor{brown}{D}_3 \Psi_2 &= -\chi_0 \dot{\Phi}_4 \\
&= \frac{1}{3}(-u_1 + 2u_2 + 2u_3)\ddot{h}_{11} + \frac{1}{3}(2u_1 - u_2 + 2u_3)\ddot{h}_{22} \\
&\quad + \frac{1}{3}(2u_1 + 2u_2 - u_3)\ddot{h}_{33} \quad . \quad (2.15)
\end{aligned}$$

Our fourth iteration with a different color, $\textcolor{red}{D}_4 \Phi_4$, takes us back to where we started, Ψ_1 , and as promised forces a consistency condition when compared with the seed relation, Eq. (2.12):

$$\begin{aligned}
-\chi_0 \textcolor{red}{D}_4 \Phi_4 &= -\chi_0 i\Psi_1 \Rightarrow \\
\frac{1}{3}(-u_1 + 2u_2 + 2u_3)\dot{\psi}_{13} - \frac{1}{3}(2u_1 - u_2 + 2u_3)\dot{\psi}_{21} &= -\chi_0(-u_1\dot{\psi}_{13} + u_2\dot{\psi}_{21} + u_3\dot{\psi}_{34}) \\
-\frac{1}{3}(2u_1 + 2u_2 - u_3)\dot{\psi}_{34} & \quad (2.16)
\end{aligned}$$

This constraint simplifies to the following under the choice of cis or trans:

$$\text{cis: } \chi_0 = 1 \quad \Rightarrow \quad u_1 = u_2 = u_3 \quad , \quad (2.17)$$

$$\text{trans: } \chi_0 = -1 \quad \Rightarrow \quad u_1 + u_2 + u_3 = 0 \quad . \quad (2.18)$$

The four iterations, Eqs. (2.12) through (2.16), that lead us to the constraint Eqs. (2.17) and (2.18) can be succinctly depicted with the adinkras in Fig. 2.

Further iterations define the other four nodes of each adinkraic representation but lead to no further constraints on u_1 , u_2 , or u_3 . These resulting nodal definitions for

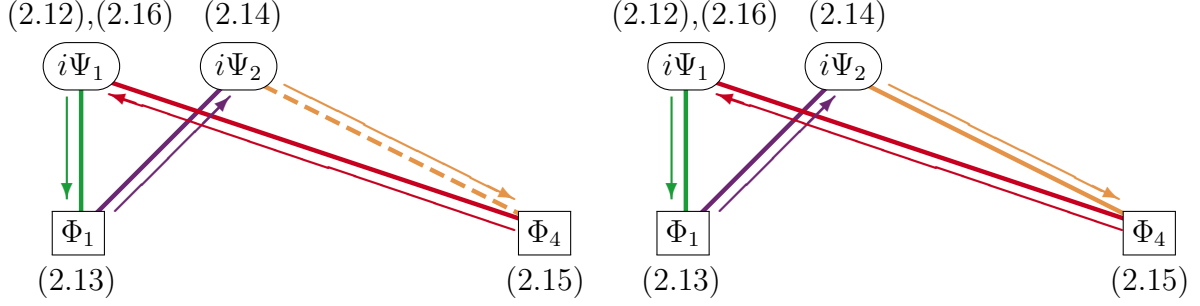


Figure 2: On the left (right), the iterative procedure in Eqs. (2.12) through (2.16) that leads to the cis (trans) constraint in Eq. (2.17). This is the cis-valise (trans-valise) adinkra in Fig. 1 with the nodes and links that do not enter the iteration removed.

the cis and trans choices are

$$\Phi = \begin{pmatrix} -u_1 A_0 \\ -u_1 P \\ u_1 S \\ -u_1 \dot{h} \end{pmatrix}, \quad i\Psi = \begin{pmatrix} -u_1(\dot{\psi}_{13} - \dot{\psi}_{21} - \dot{\psi}_{34}) \\ -u_1(-\dot{\psi}_{14} - \dot{\psi}_{22} - \dot{\psi}_{33}) \\ -u_1(\dot{\psi}_{11} + \dot{\psi}_{23} - \dot{\psi}_{32}) \\ -u_1(\dot{\psi}_{12} - \dot{\psi}_{24} + \dot{\psi}_{31}) \end{pmatrix},$$

$$\text{cis: } \chi_0 = 1 \quad , \quad u_1 \text{ unconstrained}, \quad (2.19)$$

$$\Phi = \begin{pmatrix} u_1 A_1 + (u_2 - u_3) \dot{h}_{23} \\ u_3 A_3 + (u_1 - u_2) \dot{h}_{12} \\ u_2 A_2 + (u_3 - u_1) \dot{h}_{31} \\ -u_1 \dot{h}_{11} - u_2 \dot{h}_{22} - u_3 \dot{h}_{33} \end{pmatrix}, \quad i\Psi = \begin{pmatrix} -u_1 \dot{\psi}_{13} + u_2 \dot{\psi}_{21} + u_3 \dot{\psi}_{34} \\ -u_1 \dot{\psi}_{14} - u_2 \dot{\psi}_{22} - u_3 \dot{\psi}_{33} \\ -u_1 \dot{\psi}_{11} - u_2 \dot{\psi}_{23} + u_3 \dot{\psi}_{32} \\ -u_1 \dot{\psi}_{12} + u_2 \dot{\psi}_{24} - u_3 \dot{\psi}_{31} \end{pmatrix},$$

$$\text{trans: } \chi_0 = -1 \quad , \quad u_1 + u_2 + u_3 = 0. \quad (2.20)$$

The cis choice, Eq. 2.19, has only one free parameter, u_1 , which encodes an overall scale freedom in the definition of the nodes. We can, with no loss of generality, set this parameters to $u_1 = -1$. The cis-valise submultiplet is therefore unique and so we have $n_c = 1$ for mSG. We see in Eq. (2.19) that the cis definition of Φ contains the trace of the graviton. This is related to the rotational symmetry fixed by the solution, Eq. (2.17). Notice that the two solutions (2.17) and (2.18) form a line and a plane through the origin in \mathbb{R}^3 , respectively, that are perpendicular to each other. In contrast to there being one unique cis choice, there are two linearly independent ways the trans choice, Eq. (2.20), can be satisfied. We see then that mSG has the

SUSY enantiomer numbers

$$n_c = 1 \quad , \quad n_t = 2 \quad , \quad (2.21)$$

which are the same as those for the complex linear supermultiplet [13].

The equations

$$\Phi = \left(\begin{array}{c} A_0 \\ P \\ -S \\ \hline u_1 A_1 + (u_2 - u_3) \dot{h}_{23} \\ u_3 A_3 + (u_1 - u_2) \dot{h}_{12} \\ u_2 A_2 + (u_3 - u_1) \dot{h}_{31} \\ -u_1 \dot{h}_{11} - u_2 \dot{h}_{22} - u_3 \dot{h}_{33} \\ \hline v_1 A_1 + (v_2 - v_3) \dot{h}_{23} \\ v_3 A_3 + (v_1 - v_2) \dot{h}_{12} \\ v_2 A_2 + (v_3 - v_1) \dot{h}_{31} \\ -v_1 \dot{h}_{11} - v_2 \dot{h}_{22} - v_3 \dot{h}_{33} \end{array} \right) , \quad i\Psi = \left(\begin{array}{c} \dot{\psi}_{13} - \dot{\psi}_{21} - \dot{\psi}_{34} \\ -\dot{\psi}_{14} - \dot{\psi}_{22} - \dot{\psi}_{33} \\ \dot{\psi}_{11} + \dot{\psi}_{23} - \dot{\psi}_{32} \\ \hline \dot{\psi}_{12} - \dot{\psi}_{24} + \dot{\psi}_{31} \\ \hline -u_1 \dot{\psi}_{13} + u_2 \dot{\psi}_{21} + u_3 \dot{\psi}_{34} \\ -u_1 \dot{\psi}_{14} - u_2 \dot{\psi}_{22} - u_3 \dot{\psi}_{33} \\ -u_1 \dot{\psi}_{11} - u_2 \dot{\psi}_{23} + u_3 \dot{\psi}_{32} \\ -u_1 \dot{\psi}_{12} + u_2 \dot{\psi}_{24} - u_3 \dot{\psi}_{31} \\ \hline -v_1 \dot{\psi}_{13} + v_2 \dot{\psi}_{21} + v_3 \dot{\psi}_{34} \\ -v_1 \dot{\psi}_{14} - v_2 \dot{\psi}_{22} - v_3 \dot{\psi}_{33} \\ -v_1 \dot{\psi}_{11} - v_2 \dot{\psi}_{23} + v_3 \dot{\psi}_{32} \\ -v_1 \dot{\psi}_{12} + v_2 \dot{\psi}_{24} - v_3 \dot{\psi}_{31} \end{array} \right) \quad (2.22)$$

define the nodal field content for the complete adinkraic decomposition for mSG where v_1 , v_2 , and v_3 parameterize the second trans-valise submultiplet and satisfy the same constraint as u_1 , u_2 , and u_3 :

$$u_1 + u_2 + u_3 = v_1 + v_2 + v_3 = 0 \quad (2.23)$$

Furthermore, considering our free parameters to be $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, we must also enforce linear independence: $\vec{v} \not\propto \vec{u}$, otherwise they would be describing the exact same field content up to an overall rescaling.

The node definitions (2.22) collapse Tables 1 and 2 into Table 4 three different ways, which is encoded in the valise-adinkra for the full mSG multiplet in Fig. 3. In Section 4, we will see that the cSG adinkra is composed precisely of the two trans-valise submultiplets of mSG. We identify the cis-valise submultiplet of mSG as the chiral compensator, σ , as explained in Sec. 1. This is consistent with prior results [12, 13] that found the chiral multiplet to be the cis-adinkra in valise form, with SUSY enantiomer numbers $n_c = 1$, $n_t = 0$. We can lower all fermionic nodes and the node that contains the trace of the graviton $h = h_{11} + h_{22} + h_{33}$, and swap a sign on S and $i\Psi_2 = -\dot{\psi}_{14} - \dot{\psi}_{22} - \dot{\psi}_{33}$ to arrive at the extended adinkra in Fig. 4. Notice the nodal field content is now without derivatives. The trans-valise adinkras do not have this capability as all of the allowed choices for \vec{u} and \vec{v} will lead to nodal field definitions that are linear combinations of fields that are of different engineering dimensions.

Table 4: Zero-brane reduced mSG transformation rules in the adinkraic representation in Eqs. (2.22).

	D ₁	D ₂	D ₃	D ₄		D ₁	D ₂	D ₃	D ₄
Φ_1	$i\Psi_1$	$i\Psi_2$	$i\Psi_3$	$-i\Psi_4$	Ψ_1	Φ_1	$-\Phi_2$	$-\Phi_3$	Φ_4
Φ_2	$i\Psi_2$	$-i\Psi_1$	$i\Psi_4$	$i\Psi_3$	Ψ_2	Φ_2	Φ_1	$-\Phi_4$	$-\Phi_3$
Φ_3	$i\Psi_3$	$-i\Psi_4$	$-i\Psi_1$	$-i\Psi_2$	Ψ_3	Φ_3	Φ_4	Φ_1	Φ_2
Φ_4	$i\Psi_4$	$i\Psi_3$	$-i\Psi_2$	$i\Psi_1$	Ψ_4	Φ_4	$-\Phi_3$	Φ_2	$-\Phi_1$
Φ_5	$i\Psi_5$	$i\Psi_6$	$-i\Psi_7$	$-i\Psi_8$	Ψ_5	Φ_5	$-\Phi_6$	Φ_7	Φ_8
Φ_6	$i\Psi_6$	$-i\Psi_5$	$-i\Psi_8$	$i\Psi_7$	Ψ_6	Φ_6	Φ_5	Φ_8	$-\Phi_7$
Φ_7	$i\Psi_7$	$-i\Psi_8$	$i\Psi_5$	$-i\Psi_6$	Ψ_7	Φ_7	Φ_8	$-\Phi_5$	Φ_6
Φ_8	$i\Psi_8$	$i\Psi_7$	$i\Psi_6$	$i\Psi_5$	Ψ_8	Φ_8	$-\Phi_7$	$-\Phi_6$	$-\Phi_5$
Φ_9	$i\Psi_9$	$i\Psi_{10}$	$-i\Psi_{11}$	$-i\Psi_{12}$	Ψ_9	Φ_9	$-\Phi_{10}$	Φ_{11}	Φ_{12}
Φ_{10}	$-i\Psi_9$	$-i\Psi_{10}$	$-i\Psi_{12}$	$i\Psi_{11}$	Ψ_{10}	Φ_{10}	Φ_9	Φ_{12}	$-\Phi_{11}$
Φ_{11}	$i\Psi_{11}$	$-i\Psi_{12}$	$i\Psi_9$	$-i\Psi_{10}$	Ψ_{11}	Φ_{11}	Φ_{12}	$-\Phi_9$	Φ_{10}
Φ_{12}	$i\Psi_{12}$	$i\Psi_{11}$	$i\Psi_{10}$	$i\Psi_9$	Ψ_{12}	Φ_{12}	$-\Phi_{11}$	$-\Phi_{10}$	$-\Phi_9$

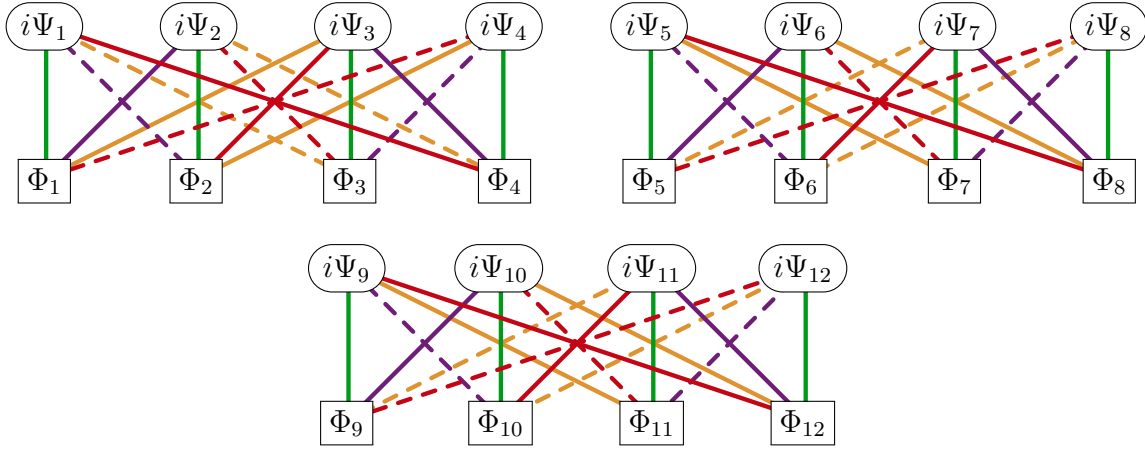


Figure 3: The mSG valise adinkra is composed of one cis-valise adinkra (upper left) and two trans-valise adinkras (upper right and bottom). The mSG multiplet therefore has SUSY enantiomer numbers $n_c = 1$, $n_t = 2$. This is the exact same valise adinkra found for the complex linear supermultiplet in Ref. [13]. The engineering dimensions of all bosons are the same and the engineering dimensions of all fermions are the same.

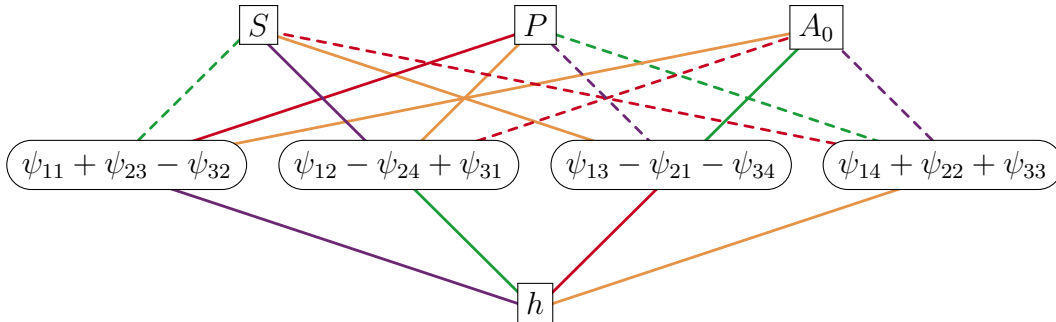


Figure 4: The chiral compensator submultiplet of mSG. It is the cis-valise adinkra in Fig. 3 with all fermion nodes lowered and the graviton trace node lowered. Also, the S node and $\psi_{14} + \psi_{22} + \psi_{33}$ nodes have the opposite sign as in the cis-valise.

Table 4 can be succinctly written as Eq. (1.1) with adinkra matrices

$$\begin{aligned} \mathbf{L}_1 &= \mathbf{I}_3 \otimes \mathbf{I}_4, & \mathbf{L}_2 &= i\mathbf{I}_3 \otimes \boldsymbol{\beta}_3, \\ \mathbf{L}_3 &= i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \boldsymbol{\beta}_2, & \mathbf{L}_4 &= -i\mathbf{I}_3 \otimes \boldsymbol{\beta}_1. \end{aligned} \quad (2.24)$$

We have written the adinkra matrices (2.24) in terms of the $SO(4)$ generators

$$\begin{aligned} i\boldsymbol{\alpha}^1 &= i\boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma}^1, & i\boldsymbol{\alpha}^2 &= i\mathbf{I}_2 \otimes \boldsymbol{\sigma}^2, & i\boldsymbol{\alpha}^3 &= i\boldsymbol{\sigma}^2 \otimes \boldsymbol{\sigma}^3, \\ i\boldsymbol{\beta}^1 &= i\boldsymbol{\sigma}^1 \otimes \boldsymbol{\sigma}^2, & i\boldsymbol{\beta}^2 &= i\boldsymbol{\sigma}^2 \otimes \mathbf{I}_2, & i\boldsymbol{\beta}^3 &= i\boldsymbol{\sigma}^3 \otimes \boldsymbol{\sigma}^2. \end{aligned} \quad (2.25)$$

The adinkra matrices (2.24) satisfy the orthogonal relationship

$$\mathbf{R}_I = \mathbf{L}_I^t = \mathbf{L}_I^{-1}, \quad (2.26)$$

and the garden algebra

$$\begin{aligned} \mathbf{L}_I \mathbf{R}_J + \mathbf{L}_J \mathbf{R}_I &= 2\delta_{IJ} \mathbf{I}_{12} \\ \mathbf{R}_I \mathbf{L}_J + \mathbf{R}_J \mathbf{L}_I &= 2\delta_{IJ} \mathbf{I}_{12} \end{aligned} \quad (2.27)$$

with \mathbf{I}_n the $n \times n$ identity matrix. These are identical to those for the complex linear supermultiplet as presented in Ref. [13].

2.3 Traces

The chromocharacters, as defined in Refs. [12, 13], are

$$\begin{aligned} Tr[\mathbf{L}_I \mathbf{L}_J^t] &= 4(n_c + n_t) \delta_{IJ} \\ Tr[\mathbf{L}_I \mathbf{L}_J^t \mathbf{L}_K \mathbf{L}_L^t] &= 4(n_c + n_t)(\delta_{IJ}\delta_{KL} - \delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}) \\ &\quad + 4(n_c - n_t) \epsilon_{IJKL}. \end{aligned} \quad (2.28)$$

The chromocharacters for mSG are

$$\begin{aligned} Tr[\mathbf{L}_I \mathbf{L}_J^t] &= 12 \delta_{IJ} \\ Tr[\mathbf{L}_I \mathbf{L}_J^t \mathbf{L}_K \mathbf{L}_L^t] &= 12(\delta_{IJ}\delta_{KL} - \delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}) \\ &\quad - 4 \epsilon_{IJKL}, \end{aligned} \quad (2.29)$$

the same as for the complex linear supermultiplet [13], with SUSY enantiomer numbers $n_c = 1, n_t = 2$. The top left adinkra in the full valise of Fig. 3 is the cis-adinkra, and the top right and bottom adinkras are the two trans-adinkras. As explained in

Sec. 1, at the level of representations, mSG is a chiral compensator superfield added to cSG. Since we have already identified the cis-adinkra in Fig. 4 as the (4|4) chiral compensator, we would be led to believe that the two trans-adinkras compose the (8|8) cSG. We will indeed find this to be true in Section 4.

3 4D, $\mathcal{N} = 1$ Non-Minimal SG ($n \neq -1/3, 0$)

The linearized theory of 4D, $\mathcal{N} = 1$ Poincaré non-minimal supergravity (mSG) contains the real component fields of a scalar auxiliary field \tilde{S} , pseudoscalar auxiliary field \tilde{P} , axial vector auxiliary field \tilde{A}_μ , Majorana gravitino $\psi_{\mu a}$, graviton $h_{\mu\nu}$, vector auxiliary field \tilde{V}_μ , axial vector auxiliary field \tilde{W}_μ , and Majorana auxiliary fermions $\tilde{\lambda}_a$ and $\tilde{\beta}_a$ [8].

3.1 Transformation Laws

To proceed with adinkra analysis, we must write the transformation laws for mSG [8] in Majorana notation. These transformation laws depend on the parameter $n \neq -1/3, 0$ and are given by

$$D_a \tilde{S} = \frac{1}{4N}([\gamma^\mu, \gamma^\nu])_a{}^d \partial_\mu \psi_{\nu d} - \frac{nN}{4} \tilde{\beta}_a + (\gamma^\nu)_a{}^d \partial_\nu \tilde{\lambda}_d \quad (3.1a)$$

$$D_a \tilde{P} = i \frac{1}{4N}(\gamma^5[\gamma^\mu, \gamma^\nu])_a{}^d \partial_\mu \psi_{\nu d} - i \frac{nN}{4}(\gamma^5)_a{}^d \tilde{\beta}_d + i(\gamma^5 \gamma^\nu)_a{}^d \partial_\nu \tilde{\lambda}_d \quad (3.1b)$$

$$D_a \tilde{A}_\mu = -i \frac{1}{2}(\gamma^5 \gamma^\nu)_a{}^d \partial_{[\nu} \psi_{\mu]d} + \frac{1}{4} \epsilon_\mu{}^{\nu\alpha\beta} (\gamma_\nu)_a{}^d \partial_\alpha \psi_{\beta d} + i 2N (\gamma^5)_a{}^d \partial_\mu \tilde{\lambda}_d \quad (3.1c)$$

$$D_a h_{\mu\nu} = \frac{1}{2} (\gamma_{(\mu})_a{}^d \psi_{\nu)d} \quad (3.1d)$$

$$D_a \tilde{W}_\mu = i \frac{nN}{4} (\gamma^5 \gamma_\mu)_a{}^d \tilde{\beta}_d - i \frac{2N}{3} (\gamma^5)_a{}^d \partial_\mu \tilde{\lambda}_d - i \frac{1}{6} (\gamma^5 \gamma^\nu)_a{}^d \partial_{[\mu} \psi_{\nu]d} + \\ + i (\gamma^5 \gamma^\nu \gamma_\mu)_a{}^d \partial_\nu \tilde{\lambda}_d + \frac{1}{6} \epsilon_\mu{}^{\nu\alpha\beta} (\gamma_\beta)_a{}^d \partial_\nu \psi_{\alpha d} \quad (3.1e)$$

$$D_a \tilde{V}_\mu = -\frac{nN}{4} (\gamma_\mu)_a{}^d \tilde{\beta}_d + (\gamma^\nu \gamma_\mu)_a{}^d \partial_\nu \tilde{\lambda}_d \quad (3.1f)$$

$$D_a \psi_{\mu c} = -i 2nN (\gamma_\mu)_{ac} \tilde{S} + 2nN (\gamma^5 \gamma_\mu)_{ac} \tilde{P} - 2(\gamma^5)_{ac} \tilde{A}_\mu + \frac{2}{3} (\gamma^5 \gamma^\nu \gamma_\mu)_{ac} \tilde{A}_\nu \\ - i \frac{1}{2} ([\gamma^\alpha, \gamma^\beta])_{ac} \partial_\alpha h_{\beta\mu} + 2nN (\gamma^5 \gamma^\nu \gamma_\mu)_{ac} \tilde{W}_\nu \quad (3.1g)$$

$$D_a \tilde{\lambda}_c = -i \frac{nN}{2} C_{ac} \tilde{S} + \frac{nN}{2} (\gamma^5)_{ac} \tilde{P} + \frac{3n}{2} (\gamma^5 \gamma^\mu)_{ac} \tilde{W}_\mu + i \frac{1}{2} (\gamma^\mu)_{ac} \tilde{V}_\mu \quad (3.1h)$$

$$D_a \tilde{\beta}_c = -i 2(\gamma^\mu)_{ac} \partial_\mu \tilde{S} + 2(\gamma^5 \gamma^\mu)_{ac} \partial_\mu \tilde{P} - \frac{3}{N} (\gamma^5 [\gamma^\mu, \gamma^\nu])_{ac} \partial_\mu \tilde{W}_\nu + i \frac{2}{nN} C_{ac} \partial^\mu \tilde{V}_\mu \\ + \frac{4}{3nN} (\gamma^5)_{ac} \partial_\mu \tilde{A}^\mu + 2 \frac{nN+1}{nN} (\gamma^5)_{ac} \partial^\mu \tilde{W}_\mu + i \frac{1}{nN} ([\gamma^\mu, \gamma^\nu])_{ac} \partial_\mu \tilde{V}_\nu \quad (3.1i)$$

where

$$N = \frac{3n+1}{n} \quad . \quad (3.2)$$

These are an invariance of the same Lagrangian as was presented in Ref. [8] in the linearized limit

$$\begin{aligned} \mathcal{L}_{\text{mSG}} = & -\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \frac{1}{2}\partial_\alpha h\partial^\alpha h - \partial^\alpha h\partial^\beta h_{\alpha\beta} + \partial^\mu h_{\mu\nu}\partial_\alpha h^{\alpha\nu} + \frac{4}{3}\tilde{A}_\mu\tilde{A}^\mu \\ & + \frac{4(3n+1)^2}{n}(\tilde{S}^2 + \tilde{P}^2) - \frac{4(3n+1)}{n}\tilde{V}_\mu\tilde{V}^\mu - 12(3n+1)\tilde{W}_\mu\tilde{W}^\mu \\ & - \frac{1}{2}\psi_{\mu a}\epsilon^{\mu\nu\alpha\beta}(\gamma^5\gamma_\nu)^{ab}\partial_\alpha\psi_{\beta b} - i\frac{4(3n+1)^2}{n}C^{ab}\tilde{\lambda}_a\tilde{\beta}_b \quad . \end{aligned} \quad (3.3)$$

The algebra closes on the auxiliary fields $\tilde{X} = (\tilde{S}, \tilde{P}, \tilde{A}_\mu, \tilde{V}_\mu, \tilde{W}_\mu, \tilde{\lambda}_a, \tilde{\beta}_a)$ as

$$\{D_a, D_b\}\tilde{X} = 2i(\gamma^\mu)_{ab}\partial_\mu\tilde{X} \quad (3.4)$$

and on the physical fields $\psi_{\mu a}$ and $h_{\mu\nu}$ as

$$\begin{aligned} \{D_a, D_b\}h_{\mu\nu} = & 2i(\gamma^\alpha)_{ab}\partial_\alpha h_{\mu\nu} - i(\gamma^\alpha)_{ab}\partial_{(\mu}h_{\nu)\alpha} \\ \{D_a, D_b\}\psi_{\mu c} = & 2i(\gamma^\alpha)_{ab}\partial_\alpha\psi_{\mu c} - i\partial_\mu\varphi_{abc} - i\partial_\mu\chi_{abc} \end{aligned} \quad (3.5)$$

where φ_{abc} is the same as for the minimal case, Eq. (2.6). The new term on the right hand side of the gravitino algebra

$$\chi_{abc} = \frac{2(3n+1)}{8n} \left(8(\gamma^\alpha)_{ab}(\gamma_\alpha)_c{}^d + [\gamma^\alpha, \gamma^\beta]_{ab}[\gamma_\alpha, \gamma_\beta]_c{}^d \right) \tilde{\lambda}_d \quad (3.6)$$

is in terms of the new auxiliary fermion λ_a and is also a consequence of the gauge symmetry Eq. (2.8) of the Lagrangian.

To facilitate finding the mSG adinkra, it will be advantageous to move to a basis in the $4D, \mathcal{N} = 1$ theory where the transformation laws take a simpler form. Performing the following field redefinitions

$$\beta_a = \frac{n}{2}[\gamma^\mu, \gamma^\nu]_a{}^d\partial_\mu\psi_{\nu d} - \frac{(3n+1)^2}{2}\tilde{\beta}_a + 2\partial_\nu\tilde{\lambda}_a \quad , \quad (3.7a)$$

$$\lambda_a = 4\frac{3n+1}{n}\tilde{\lambda}_a \quad , \quad (3.7b)$$

$$S = 2(3n+1)\tilde{S} \quad , \quad P = 2(3n+1)\tilde{P} \quad , \quad (3.7c)$$

$$W_\mu = 2(3n+1)\tilde{W}_\mu + \frac{2}{3}\tilde{A}_\mu \quad , \quad (3.7d)$$

$$V_\mu = 2\frac{3n+1}{n}\tilde{V}_\mu \quad , \quad A_\mu = -2\tilde{A}_\mu \quad , \quad (3.7e)$$

the transformation laws become

$$D_a S = \beta_a \quad (3.8a)$$

$$D_a P = i(\gamma^5)_a{}^d \beta_d \quad (3.8b)$$

$$D_a A_\mu = i(\gamma^5 \gamma^\nu)_a{}^d \partial_{[\nu} \psi_{\mu]d} - \frac{1}{2} \epsilon_\mu{}^{\nu\alpha\beta} (\gamma_\nu)_a{}^d \partial_\alpha \psi_{\beta d} - i(\gamma^5)_a{}^d \partial_\mu \lambda_d \quad (3.8c)$$

$$D_a h_{\mu\nu} = \frac{1}{2} (\gamma_{(\mu})_a{}^d \psi_{\nu)d} \quad (3.8d)$$

$$D_a W_\mu = -i(\gamma^5 \gamma_\mu)_a{}^d \beta_d + \frac{1}{2} \epsilon_\mu{}^{\nu\alpha\beta} (\gamma_\beta)_a{}^d \partial_\nu \psi_{\alpha d} \quad (3.8e)$$

$$D_a V_\mu = n^{-1} (\gamma_\mu)_a{}^d \beta_d - \partial_\mu \lambda_a - (\gamma^\nu)_a{}^d \partial_{[\mu} \psi_{\nu]d} + (\gamma^\nu \gamma_\mu)_a{}^d \partial_\nu \lambda_d - i \epsilon_\mu{}^{\nu\alpha\beta} (\gamma^5 \gamma_\beta)_a{}^d \partial_\nu \psi_{\alpha d} \quad (3.8f)$$

$$D_a \psi_{\mu c} = -i(\gamma_\mu)_{ac} S + (\gamma^5 \gamma_\mu)_{ac} P + (\gamma^5)_{ac} A_\mu - i \frac{1}{2} ([\gamma^\alpha, \gamma^\beta])_{ac} \partial_\alpha h_{\beta\mu} + (\gamma^5 \gamma^\nu \gamma_\mu)_{ac} W_\nu \quad (3.8g)$$

$$D_a \lambda_c = -i N C_{ac} S + N (\gamma^5)_{ac} P + 3(\gamma^5 \gamma^\mu)_{ac} W_\mu + i(\gamma^\mu)_{ac} V_\mu + (\gamma^5 \gamma^\mu)_{ac} A_\mu \quad (3.8h)$$

$$D_a \beta_c = i(\gamma^\mu)_{ac} \partial_\mu S - (\gamma^5 \gamma^\mu)_{ac} \partial_\mu P - (\gamma^5)_{ac} \partial^\mu W_\mu - i n C_{ac} \partial^\mu V_\mu + i C_{ac} \partial^\mu \partial_{[\nu} h_{\mu]}{}^\nu \quad (3.8i)$$

The Lagrangian that is invariant with respect to these simplified transformation laws is:

$$\begin{aligned} \mathcal{L}_{\text{MSG}} = & -\frac{1}{2} \partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} + \frac{1}{2} \partial_\alpha h \partial^\alpha h - \partial^\alpha h \partial^\beta h_{\alpha\beta} + \partial^\mu h_{\mu\nu} \partial_\alpha h^{\alpha\nu} + \frac{1}{n} S^2 \\ & + \frac{1}{n} P^2 - \frac{2}{3n+1} W_\mu A^\mu + \frac{n}{3n+1} A_\mu A^\mu - \frac{n}{3n+1} V_\mu V^\mu \\ & - \frac{3}{3n+1} W_\mu W^\mu - \frac{1}{2} \psi_{\mu a} \epsilon^{\mu\nu\alpha\beta} (\gamma^5 \gamma_\nu)^{ab} \partial_\alpha \psi_{\beta b} + i \frac{2}{3n+1} \lambda_a \beta^a \\ & - i \frac{n}{3n+1} \lambda_b (\gamma^\mu \gamma^\nu)^{bc} \partial_{[\mu} \psi_{\nu]c} - i \frac{n}{3n+1} \lambda_b (\gamma^\nu)^{bc} \partial_\nu \lambda_c \quad . \end{aligned} \quad (3.9)$$

3.2 One-Dimensional Reduction

Once more we work in the temporal gauge (2.9) for the graviton and gravitino so that the Lagrangian (3.9) reduced to the 0-brane becomes

$$\begin{aligned}
\mathcal{L}_{\text{mSG}}^{(0)} = & \dot{h}_{12}^2 + \dot{h}_{13}^2 + \dot{h}_{23}^2 - \dot{h}_{11}\dot{h}_{22} - \dot{h}_{11}\dot{h}_{33} - \dot{h}_{22}\dot{h}_{33} + \frac{1}{n}S^2 + \frac{1}{n}P^2 \\
& + \frac{1}{3n+1} \left(2(-W_1A_1 - W_2A_2 - W_3A_3 + W_0A_0) \right. \\
& \quad + n(A_1A_1 + A_2A_2 + A_3A_3 - A_0A_0) \\
& \quad - n(V_1V_1 + V_2V_2 + V_3V_3 - V_0V_0) \\
& \quad \left. - 3(W_1W_1 + W_2W_2 + W_3W_3 - W_0W_0) \right) \\
& + i \left(-\psi_{31}\dot{\psi}_{12} + \psi_{32}\dot{\psi}_{11} - \psi_{33}\dot{\psi}_{14} + \psi_{34}\dot{\psi}_{13} - \psi_{11}\dot{\psi}_{23} + \psi_{12}\dot{\psi}_{24} \right. \\
& \quad \left. + \psi_{13}\dot{\psi}_{21} - \psi_{14}\dot{\psi}_{22} - \psi_{21}\dot{\psi}_{34} - \psi_{22}\dot{\psi}_{33} + \psi_{23}\dot{\psi}_{32} + \psi_{24}\dot{\psi}_{31} \right) \\
& + i \frac{1}{3n+1} (2(\lambda_2\beta_1 - \lambda_1\beta_2 - \lambda_4\beta_3 + \lambda_3\beta_4) - n(\lambda_1\dot{l}_1 + \lambda_2\dot{l}_2 + \lambda_3\dot{l}_3 + \lambda_4\dot{l}_4)) \\
& - i \frac{2n}{3n+1} \left(\lambda_2\dot{\psi}_{11} + \lambda_1\dot{\psi}_{12} + \lambda_4\dot{\psi}_{13} + \lambda_3\dot{\psi}_{14} - \lambda_4\dot{\psi}_{21} + \lambda_3\dot{\psi}_{22} \right. \\
& \quad \left. + \lambda_2\dot{\psi}_{23} - \lambda_1\dot{\psi}_{24} + \lambda_1\dot{\psi}_{31} - \lambda_2\dot{\psi}_{32} + \lambda_3\dot{\psi}_{33} - \lambda_4\dot{\psi}_{34} \right) \quad . \tag{3.10}
\end{aligned}$$

The 0-brane reduced transformation laws can be succinctly displayed as in Tables 5 and 6. Following the same iterative procedure depicted in Fig. 2 leads us again to one unique cis-valise submultiplet, and an infinite set of choices for trans-valise submultiplets, the nodal content of which are

$$\Phi = \begin{pmatrix} W_0 \\ P \\ S \\ n(\dot{h} + V_0) \end{pmatrix}, \quad i\Psi = \begin{pmatrix} -\beta_3 \\ -\beta_4 \\ \beta_1 \\ -\beta_2 \end{pmatrix},$$

$$\text{cis: } \chi_0 = 1 \quad , \tag{3.11}$$

$$\Phi = \begin{pmatrix} u_1A_1 + (u_2 - u_3)\dot{h}_{23} + \frac{u_4}{N} \left(A_0 + A_1 - \frac{W_0}{n} - \frac{W_1}{n} \right) - u_5W_1 \\ u_3A_3 + (u_1 - u_2)\dot{h}_{12} + \frac{u_4}{N} \left(A_3 - \frac{W_3}{n} - V_2 \right) - u_5W_3 \\ u_2A_2 + (u_3 - u_1)\dot{h}_{31} + \frac{u_4}{N} \left(A_2 - \frac{W_2}{n} + V_3 \right) - u_5W_2 \\ -u_1\dot{h}_{11} - u_2\dot{h}_{22} - u_3\dot{h}_{33} + \frac{u_4}{N}(V_0 + V_1) + u_5n(\dot{h} + V_0) \end{pmatrix},$$

$$i\Psi = \begin{pmatrix} -u_1\dot{\psi}_{13} + u_2\dot{\psi}_{21} + u_3\dot{\psi}_{34} + \frac{u_4}{N}\dot{\lambda}_4 - u_5\beta_3 \\ -u_1\dot{\psi}_{14} - u_2\dot{\psi}_{22} - u_3\dot{\psi}_{33} + 2\frac{u_4}{N}\left(\frac{\beta_4}{n} - \frac{\dot{\lambda}_3}{2} - \dot{\psi}_{14} - \dot{\psi}_{22} - \dot{\psi}_{33}\right) + u_5\beta_4 \\ -u_1\dot{\psi}_{11} - u_2\dot{\psi}_{23} + u_3\dot{\psi}_{32} + 2\frac{u_4}{N}\left(\frac{\beta_1}{n} - \frac{\dot{\lambda}_2}{2} - \dot{\psi}_{11} - \dot{\psi}_{23} + \dot{\psi}_{32}\right) + u_5\beta_1 \\ -u_1\dot{\psi}_{12} + u_2\dot{\psi}_{24} - u_3\dot{\psi}_{31} + \frac{u_4}{N}\dot{\lambda}_1 - u_5\beta_2 \end{pmatrix},$$

$$\text{trans: } \chi_0 = -1 \quad , \quad u_1 + u_2 + u_3 + u_4 + u_5 = 0. \quad (3.12)$$

Table 5: $\mathcal{H}SG$ bosonic transformation laws in temporal gauge, Eq.(2.9), and reduced to the 0-brane.

	D ₁	D ₂	D ₃	D ₄
h_{11}	ψ_{12}	ψ_{11}	ψ_{14}	ψ_{13}
h_{22}	$-\psi_{24}$	ψ_{23}	ψ_{22}	$-\psi_{21}$
h_{33}	ψ_{31}	$-\psi_{32}$	ψ_{33}	$-\psi_{34}$
V_0	$\psi_{24} - \psi_{31} - \psi_{12} - \frac{\beta_2}{n}$	$-\psi_{11} - \psi_{23} + \psi_{32} + \frac{\beta_1}{n}$	$-\psi_{14} - \psi_{22} - \psi_{33} + \frac{\beta_4}{n}$	$-\psi_{13} + \psi_{21} + \psi_{34} - \frac{\beta_3}{n}$
V_1	$\psi_{12} - \psi_{24} + \psi_{31} + \frac{\beta_2}{n} + \dot{\lambda}_1$	$\psi_{32} - \psi_{11} - \psi_{23} + \frac{\beta_1}{n} - \dot{\lambda}_2$	$\frac{\beta_4}{n} - \dot{\lambda}_3 - \psi_{14} - \psi_{22} - \psi_{33}$	$\psi_{13} - \psi_{21} - \psi_{34} + \frac{\beta_3}{n} + \dot{\lambda}_4$
h_{13}	$\frac{1}{2}\psi_{11} + \frac{1}{2}\psi_{32}$	$-\frac{1}{2}\psi_{12} + \frac{1}{2}\psi_{31}$	$\frac{1}{2}\psi_{13} + \frac{1}{2}\psi_{34}$	$-\frac{1}{2}\psi_{14} + \frac{1}{2}\psi_{33}$
S	β_1	β_2	β_3	β_4
W_2	$\frac{1}{2}(\psi_{32} - \psi_{11}) - \beta_1$	$\frac{1}{2}(\psi_{12} + \psi_{31}) - \beta_2$	$\frac{1}{2}(\psi_{34} - \psi_{13}) + \beta_3$	$\frac{1}{2}(\psi_{14} + \psi_{33}) + \beta_4$
V_3	$\frac{\beta_1}{n} - \dot{\lambda}_2 - \psi_{11} - \psi_{23} + \psi_{32}$	$\psi_{24} - \psi_{12} - \psi_{31} - \frac{\beta_2}{n} - \dot{\lambda}_1$	$\frac{\beta_3}{n} + \dot{\lambda}_4 + \psi_{13} - \psi_{21} - \psi_{34}$	$\psi_{14} + \psi_{22} + \psi_{33} - \frac{\beta_4}{n} + \dot{\lambda}_3$
A_2	$-\frac{1}{2}(-\psi_{11} + 2\psi_{23} + \psi_{32})$	$-\frac{1}{2}(\psi_{12} + 2\psi_{24} + \psi_{31})$	$-\frac{1}{2}(-\psi_{13} - 2\psi_{21} + \psi_{34})$	$-\frac{1}{2}(\psi_{14} - 2\psi_{22} + \psi_{33})$
h_{12}	$-\frac{1}{2}\psi_{14} + \frac{1}{2}\psi_{22}$	$\frac{1}{2}\psi_{13} + \frac{1}{2}\psi_{21}$	$\frac{1}{2}\psi_{12} + \frac{1}{2}\psi_{24}$	$-\frac{1}{2}\psi_{11} + \frac{1}{2}\psi_{23}$
P	$-\beta_4$	β_3	$-\beta_2$	β_1
W_3	$\frac{1}{2}(-\psi_{14} - \psi_{22}) - \beta_4$	$\frac{1}{2}(\psi_{13} - \psi_{21}) - \beta_3$	$\frac{1}{2}(\psi_{12} - \psi_{24}) - \beta_2$	$\frac{1}{2}(-\psi_{11} - \psi_{23}) - \beta_1$
V_2	$\psi_{14} + \psi_{22} + \psi_{33} - \frac{\beta_4}{n} + \dot{\lambda}_3$	$\psi_{13} - \psi_{21} - \psi_{34} + \frac{\beta_3}{n} + \dot{\lambda}_4$	$\psi_{12} - \psi_{24} + \psi_{31} + \frac{\beta_2}{n} + \dot{\lambda}_1$	$\psi_{11} + \psi_{23} - \psi_{32} - \frac{\beta_1}{n} + \dot{\lambda}_2$
A_3	$-\frac{1}{2}(-\psi_{14} - \psi_{22} + 2\psi_{33})$	$-\frac{1}{2}(\psi_{13} - \psi_{21} + 2\psi_{34})$	$-\frac{1}{2}(\psi_{12} - \psi_{24} - 2\psi_{31})$	$-\frac{1}{2}(-\psi_{11} - \psi_{23} - 2\psi_{32})$
h_{23}	$\frac{1}{2}\psi_{21} - \frac{1}{2}\psi_{34}$	$-\frac{1}{2}\psi_{22} + \frac{1}{2}\psi_{33}$	$\frac{1}{2}\psi_{23} + \frac{1}{2}\psi_{32}$	$-\frac{1}{2}\psi_{24} - \frac{1}{2}\psi_{31}$
W_0	$-\beta_3$	$-\beta_4$	β_1	β_2
W_1	$\frac{1}{2}(\psi_{21} + \psi_{34}) + \beta_3$	$\frac{1}{2}(-\psi_{22} - \psi_{33}) - \beta_4$	$\frac{1}{2}(\psi_{23} - \psi_{32}) + \beta_1$	$\frac{1}{2}(\psi_{31} - \psi_{24}) - \beta_2$
A_0	$\psi_{13} - \psi_{21} - \psi_{34} + \dot{\lambda}_4$	$-\psi_{14} - \psi_{22} - \psi_{33} - \dot{\lambda}_3$	$\psi_{11} + \psi_{23} - \psi_{32} + \dot{\lambda}_2$	$-\psi_{12} + \psi_{24} - \psi_{31} - \dot{\lambda}_1$
A_1	$-\frac{1}{2}(2\psi_{13} + \psi_{21} + \psi_{34})$	$-\frac{1}{2}(2\psi_{14} - \psi_{22} - \psi_{33})$	$-\frac{1}{2}(-2\psi_{11} + \psi_{23} - \psi_{32})$	$-\frac{1}{2}(-2\psi_{12} - \psi_{24} + \psi_{31})$

Table 6: $\mathcal{H}SG$ fermionic transformation laws in temporal gauge, Eq.(2.9), and reduced to the 0-brane.

	D ₁	D ₂	D ₃	D ₄
ψ_{11}	$i\dot{h}_{13} - iS - iW_2$	$i\dot{h}_{11}$	$iA_1 - iW_0 + iW_1$	$-i\dot{h}_{12} - iP - iW_3$
ψ_{23}	$-iA_2 - iS - iW_2$	$i\dot{h}_{22}$	$i\dot{h}_{23} - iW_0 + iW_1$	$i\dot{h}_{12} - iP - iW_3$
ψ_{32}	$i\dot{h}_{13} + iS + iW_2$	$-i\dot{h}_{33}$	$i\dot{h}_{23} + iW_0 - iW_1$	$iA_3 + iP + iW_3$
λ_2	$iA_2 + iNS - iV_3 + 3iW_2$	$i(V_0 - V_1)$	$iA_0 - iA_1 + 3iW_0 - 3iW_1$	$iA_3 + iNP + iV_2 + 3iW_3$
β_1	$i\dot{S}$	$in(\dot{h} + \dot{V}_0)$	$i\dot{W}_0$	$i\dot{P}$
ψ_{12}	$i\dot{h}_{11}$	$-i\dot{h}_{13} + iS + iW_2$	$i\dot{h}_{12} - iP + iW_3$	$iA_1 + iW_0 + iW_1$
ψ_{24}	$-i\dot{h}_{22}$	$-iA_2 - iS - iW_2$	$i\dot{h}_{12} + iP - iW_3$	$-i\dot{h}_{23} - iW_0 - iW_1$
ψ_{31}	$i\dot{h}_{33}$	$i\dot{h}_{13} + iS + iW_2$	$iA_3 - iP + iW_3$	$-i\dot{h}_{23} + iW_0 + iW_1$
λ_1	$i(V_0 + V_1)$	$-iA_2 - iNS - iV_3 - 3iW_2$	$-iA_3 + iNP + iV_2 - 3iW_3$	$-iA_0 - iA_1 - 3iW_0 - 3iW_1$
β_2	$-in(\dot{h} + \dot{V}_0)$	$i\dot{S}$	$-i\dot{P}$	$i\dot{W}_0$
ψ_{13}	$-iA_1 - iW_0 - iW_1$	$i\dot{h}_{12} + iP + iW_3$	$i\dot{h}_{13} + iS - iW_2$	$i\dot{h}_{11}$
ψ_{21}	$i\dot{h}_{23} + iW_0 + iW_1$	$i\dot{h}_{12} - iP - iW_3$	$iA_2 - iS + iW_2$	$-i\dot{h}_{22}$
ψ_{34}	$-i\dot{h}_{23} + iW_0 + iW_1$	$-iA_3 - iP - iW_3$	$i\dot{h}_{13} - iS + iW_2$	$-i\dot{h}_{33}$
λ_4	$iA_0 + iA_1 + 3iW_0 + 3iW_1$	$-iA_3 - iNP + iV_2 - 3iW_3$	$iA_2 - iNS + iV_3 + 3iW_2$	$i(V_0 + V_1)$
β_3	$-i\dot{W}_0$	$i\dot{P}$	$i\dot{S}$	$-in(\dot{h} + \dot{V}_0)$
ψ_{14}	$-i\dot{h}_{12} + iP - iW_3$	$-iA_1 + iW_0 - iW_1$	$i\dot{h}_{11}$	$-i\dot{h}_{13} - iS + iW_2$
ψ_{22}	$i\dot{h}_{12} + iP - iW_3$	$-i\dot{h}_{23} + iW_0 - iW_1$	$i\dot{h}_{22}$	$iA_2 - iS + iW_2$
ψ_{33}	$-iA_3 + iP - iW_3$	$i\dot{h}_{23} + iW_0 - iW_1$	$i\dot{h}_{33}$	$i\dot{h}_{13} - iS + iW_2$
λ_3	$iA_3 - iNP + iV_2 + 3iW_3$	$-iA_0 + iA_1 - 3iW_0 + 3iW_1$	$i(V_0 - V_1)$	$-iA_2 + iNS + iV_3 - 3iW_2$
β_4	$-i\dot{P}$	$-i\dot{W}_0$	$in(\dot{h} + \dot{V}_0)$	$i\dot{S}$

Since there is one unique cis choice, we have $n_c = 1$ for $\cancel{\mu}\text{SG}$ as was the case for mSG . Since $k = 5 = n_c + n_t$ for $\cancel{\mu}\text{SG}$ ($\cancel{\mu}\text{SG}$ is a $(20|20)$ degree of freedom multiplet, see again the discussion at the end of Section 1.1), we must chose four linearly independent trans-submultiplets to complete the adinkra and so we have $n_t = 4$ for $\cancel{\mu}\text{SG}$. In summary, the SUSY enantiomer numbers for $\cancel{\mu}\text{SG}$ are

$$n_c = 1 \quad , \quad n_t = 4 \quad . \quad (3.13)$$

To fill out the nodal field content of the $(20|20)$ $\cancel{\mu}\text{SG}$ multiplet, we must make three more copies of the trans-submultiplet, Eq. (3.12), by creating three more sets of parameters, v_i , q_i , and p_i , that are constrained exactly as the u_i are:

$$u_1 + u_2 + u_3 + u_4 + u_5 = 0 \quad (3.14a)$$

$$v_1 + v_2 + v_3 + v_4 + v_5 = 0 \quad (3.14b)$$

$$p_1 + p_2 + p_3 + p_4 + p_5 = 0 \quad (3.14c)$$

$$q_1 + q_2 + q_3 + q_4 + q_5 = 0 \quad . \quad (3.14d)$$

Furthermore, choosing the independent parameters from each set to be

$$\begin{aligned} \vec{u} &= (u_1, u_2, u_3, u_4) \quad , \quad \vec{v} = (v_1, v_2, v_3, v_4) \quad , \\ \vec{p} &= (p_1, p_2, p_3, p_4) \quad , \quad \vec{q} = (q_1, q_2, q_3, q_4) \quad , \end{aligned} \quad (3.15)$$

we have the condition that these must be linearly independent. This is, as before, so that each trans-submultiplet has unique nodal field content. The field content for all nodes of the $\cancel{\mu}\text{SG}$ adinkra is shown in Eqs. (3.16) and (3.17) where horizontal lines separate the five submultiplets. These node definitions collapse Tables 5 and 6 to Table 7 which is succinctly displayed as the valise adinkra for $\cancel{\mu}\text{SG}$ in Fig. 5.

$$\Phi = \left(\begin{array}{c} W_0 \\ P \\ S \\ n(\dot{h} + V_0) \\ \hline u_1 A_1 + (u_2 - u_3)\dot{h}_{23} + \frac{u_4}{N} \left(A_0 + A_1 - \frac{W_0}{n} - \frac{W_1}{n} \right) - u_5 W_1 \\ u_3 A_3 + (u_1 - u_2)\dot{h}_{12} + \frac{u_4}{N} \left(A_3 - \frac{W_3}{n} - V_2 \right) - u_5 W_3 \\ u_2 A_2 + (u_3 - u_1)\dot{h}_{31} + \frac{u_4}{N} \left(A_2 - \frac{W_2}{n} + V_3 \right) - u_5 W_2 \\ -u_1 \dot{h}_{11} - u_2 \dot{h}_{22} - u_3 \dot{h}_{33} + \frac{u_4}{N} (V_0 + V_1) + u_5 n(\dot{h} + V_0) \\ \hline v_1 A_1 + (v_2 - v_3)\dot{h}_{23} + \frac{v_4}{N} \left(A_0 + A_1 - \frac{W_0}{n} - \frac{W_1}{n} \right) - v_5 W_1 \\ v_3 A_3 + (v_1 - v_2)\dot{h}_{12} + \frac{v_4}{N} \left(A_3 - \frac{W_3}{n} - V_2 \right) - v_5 W_3 \\ v_2 A_2 + (v_3 - v_1)\dot{h}_{31} + \frac{v_4}{N} \left(A_2 - \frac{W_2}{n} + V_3 \right) - v_5 W_2 \\ -v_1 \dot{h}_{11} - v_2 \dot{h}_{22} - v_3 \dot{h}_{33} + \frac{v_4}{N} (V_0 + V_1) + v_5 n(\dot{h} + V_0) \\ \hline q_1 A_1 + (q_2 - q_3)\dot{h}_{23} + \frac{q_4}{N} \left(A_0 + A_1 - \frac{W_0}{n} - \frac{W_1}{n} \right) - q_5 W_1 \\ q_3 A_3 + (q_1 - q_2)\dot{h}_{12} + \frac{q_4}{N} \left(A_3 - \frac{W_3}{n} - V_2 \right) - q_5 W_3 \\ q_2 A_2 + (q_3 - q_1)\dot{h}_{31} + \frac{q_4}{N} \left(A_2 - \frac{W_2}{n} + V_3 \right) - q_5 W_2 \\ -q_1 \dot{h}_{11} - q_2 \dot{h}_{22} - q_3 \dot{h}_{33} + \frac{q_4}{N} (V_0 + V_1) + q_5 n(\dot{h} + V_0) \\ \hline p_1 A_1 + (p_2 - p_3)\dot{h}_{23} + \frac{p_4}{N} \left(A_0 + A_1 - \frac{W_0}{n} - \frac{W_1}{n} \right) - p_5 W_1 \\ p_3 A_3 + (p_1 - p_2)\dot{h}_{12} + \frac{p_4}{N} \left(A_3 - \frac{W_3}{n} - V_2 \right) - p_5 W_3 \\ p_2 A_2 + (p_3 - p_1)\dot{h}_{31} + \frac{p_4}{N} \left(A_2 - \frac{W_2}{n} + V_3 \right) - p_5 W_2 \\ -p_1 \dot{h}_{11} - p_2 \dot{h}_{22} - p_3 \dot{h}_{33} + \frac{p_4}{N} (V_0 + V_1) + p_5 n(\dot{h} + V_0) \end{array} \right) \quad (3.16)$$

$$i\Psi = \left(\begin{array}{c} -\beta_3 \\ -\beta_4 \\ \beta_1 \\ -\beta_2 \\ \hline -u_1 \dot{\psi}_{13} + u_2 \dot{\psi}_{21} + u_3 \dot{\psi}_{34} + \frac{u_4}{N} \dot{\lambda}_4 - u_5 \beta_3 \\ -u_1 \dot{\psi}_{14} - u_2 \dot{\psi}_{22} - u_3 \dot{\psi}_{33} + 2 \frac{u_4}{N} \left(\frac{\beta_4}{n} - \frac{\dot{\lambda}_3}{2} - \dot{\psi}_{14} - \dot{\psi}_{22} - \dot{\psi}_{33} \right) + u_5 \beta_4 \\ -u_1 \dot{\psi}_{11} - u_2 \dot{\psi}_{23} + u_3 \dot{\psi}_{32} + 2 \frac{u_4}{N} \left(\frac{\beta_1}{n} - \frac{\dot{\lambda}_2}{2} - \dot{\psi}_{11} - \dot{\psi}_{23} + \dot{\psi}_{32} \right) + u_5 \beta_1 \\ -u_1 \dot{\psi}_{12} + u_2 \dot{\psi}_{24} - u_3 \dot{\psi}_{31} + \frac{u_4}{N} \dot{\lambda}_1 - u_5 \beta_2 \\ \hline -v_1 \dot{\psi}_{13} + v_2 \dot{\psi}_{21} + v_3 \dot{\psi}_{34} + \frac{v_4}{N} \dot{\lambda}_4 - v_5 \beta_3 \\ -v_1 \dot{\psi}_{14} - v_2 \dot{\psi}_{22} - v_3 \dot{\psi}_{33} + 2 \frac{v_4}{N} \left(\frac{\beta_4}{n} - \frac{\dot{\lambda}_3}{2} - \dot{\psi}_{14} - \dot{\psi}_{22} - \dot{\psi}_{33} \right) + v_5 \beta_4 \\ -v_1 \dot{\psi}_{11} - v_2 \dot{\psi}_{23} + v_3 \dot{\psi}_{32} + 2 \frac{v_4}{N} \left(\frac{\beta_1}{n} - \frac{\dot{\lambda}_2}{2} - \dot{\psi}_{11} - \dot{\psi}_{23} + \dot{\psi}_{32} \right) + v_5 \beta_1 \\ -v_1 \dot{\psi}_{12} + v_2 \dot{\psi}_{24} - v_3 \dot{\psi}_{31} + \frac{v_4}{N} \dot{\lambda}_1 - v_5 \beta_2 \\ \hline -q_1 \dot{\psi}_{13} + q_2 \dot{\psi}_{21} + q_3 \dot{\psi}_{34} + \frac{q_4}{N} \dot{\lambda}_4 - q_5 \beta_3 \\ -q_1 \dot{\psi}_{14} - q_2 \dot{\psi}_{22} - q_3 \dot{\psi}_{33} + 2 \frac{q_4}{N} \left(\frac{\beta_4}{n} - \frac{\dot{\lambda}_3}{2} - \dot{\psi}_{14} - \dot{\psi}_{22} - \dot{\psi}_{33} \right) + q_5 \beta_4 \\ -q_1 \dot{\psi}_{11} - q_2 \dot{\psi}_{23} + q_3 \dot{\psi}_{32} + 2 \frac{q_4}{N} \left(\frac{\beta_1}{n} - \frac{\dot{\lambda}_2}{2} - \dot{\psi}_{11} - \dot{\psi}_{23} + \dot{\psi}_{32} \right) + q_5 \beta_1 \\ -q_1 \dot{\psi}_{12} + q_2 \dot{\psi}_{24} - q_3 \dot{\psi}_{31} + \frac{q_4}{N} \dot{\lambda}_1 - q_5 \beta_2 \\ \hline -p_1 \dot{\psi}_{13} + p_2 \dot{\psi}_{21} + p_3 \dot{\psi}_{34} + \frac{p_4}{N} \dot{\lambda}_4 - p_5 \beta_3 \\ -p_1 \dot{\psi}_{14} - p_2 \dot{\psi}_{22} - p_3 \dot{\psi}_{33} + 2 \frac{p_4}{N} \left(\frac{\beta_4}{n} - \frac{\dot{\lambda}_3}{2} - \dot{\psi}_{14} - \dot{\psi}_{22} - \dot{\psi}_{33} \right) + p_5 \beta_4 \\ -p_1 \dot{\psi}_{11} - p_2 \dot{\psi}_{23} + p_3 \dot{\psi}_{32} + 2 \frac{p_4}{N} \left(\frac{\beta_1}{n} - \frac{\dot{\lambda}_2}{2} - \dot{\psi}_{11} - \dot{\psi}_{23} + \dot{\psi}_{32} \right) + p_5 \beta_1 \\ -p_1 \dot{\psi}_{12} + p_2 \dot{\psi}_{24} - p_3 \dot{\psi}_{31} + \frac{p_4}{N} \dot{\lambda}_1 - p_5 \beta_2 \end{array} \right) \quad (3.17)$$

Table 7: Zero-brane reduced $\mathfrak{m}SG$ transformation rules in the adinkraic representation defined in Eqs. (3.16) and (3.17).

	D_1	D_2	D_3	D_4		D_1	D_2	D_3	D_4
Φ_1	$i\Psi_1$	$i\Psi_2$	$i\Psi_3$	$-i\Psi_4$	Ψ_1	Φ_1	$-\Phi_2$	$-\Phi_3$	Φ_4
Φ_2	$i\Psi_2$	$-i\Psi_1$	$i\Psi_4$	$i\Psi_3$	Ψ_2	Φ_2	Φ_1	$-\Phi_4$	$-\Phi_3$
Φ_3	$i\Psi_3$	$-i\Psi_4$	$-i\Psi_1$	$-i\Psi_2$	Ψ_3	Φ_3	Φ_4	Φ_1	Φ_2
Φ_4	$i\Psi_4$	$i\Psi_3$	$-i\Psi_2$	$i\Psi_1$	Ψ_4	Φ_4	$-\Phi_3$	Φ_2	$-\Phi_1$
Φ_5	$i\Psi_5$	$i\Psi_6$	$-i\Psi_7$	$-i\Psi_8$	Ψ_5	Φ_5	$-\Phi_6$	Φ_7	Φ_8
Φ_6	$i\Psi_6$	$-i\Psi_5$	$-i\Psi_8$	$i\Psi_7$	Ψ_6	Φ_6	Φ_5	Φ_8	$-\Phi_7$
Φ_7	$i\Psi_7$	$-i\Psi_8$	$i\Psi_5$	$-i\Psi_6$	Ψ_7	Φ_7	Φ_8	$-\Phi_5$	Φ_6
Φ_8	$i\Psi_8$	$i\Psi_7$	$i\Psi_6$	$i\Psi_5$	Ψ_8	Φ_8	$-\Phi_7$	$-\Phi_6$	$-\Phi_5$
Φ_9	$i\Psi_9$	$i\Psi_{10}$	$-i\Psi_{11}$	$-i\Psi_{12}$	Ψ_9	Φ_9	$-\Phi_{10}$	Φ_{11}	Φ_{12}
Φ_{10}	$i\Psi_{10}$	$-i\Psi_9$	$-i\Psi_{12}$	$i\Psi_{11}$	Ψ_{10}	Φ_{10}	Φ_9	Φ_{12}	$-\Phi_{11}$
Φ_{11}	$i\Psi_{11}$	$-i\Psi_{12}$	$i\Psi_9$	$-i\Psi_{10}$	Ψ_{11}	Φ_{11}	Φ_{12}	$-\Phi_9$	Φ_{10}
Φ_{12}	$i\Psi_{12}$	$i\Psi_{11}$	$i\Psi_{10}$	$i\Psi_9$	Ψ_{12}	Φ_{12}	$-\Phi_{11}$	$-\Phi_{10}$	$-\Phi_9$
Φ_{13}	$i\Psi_{13}$	$i\Psi_{14}$	$-i\Psi_{15}$	$-i\Psi_{16}$	Ψ_{13}	Φ_{13}	$-\Phi_{14}$	Φ_{15}	Φ_{16}
Φ_{14}	$i\Psi_{14}$	$-i\Psi_{13}$	$-i\Psi_{16}$	$i\Psi_{15}$	Ψ_{14}	Φ_{14}	Φ_{13}	Φ_{16}	$-\Phi_{15}$
Φ_{15}	$i\Psi_{15}$	$-i\Psi_{16}$	$i\Psi_{13}$	$-i\Psi_{14}$	Ψ_{15}	Φ_{15}	Φ_{16}	$-\Phi_{13}$	Φ_{14}
Φ_{16}	$i\Psi_{16}$	$i\Psi_{15}$	$i\Psi_{14}$	$i\Psi_{13}$	Ψ_{16}	Φ_{16}	$-\Phi_{15}$	$-\Phi_{14}$	$-\Phi_{13}$
Φ_{17}	$i\Psi_{17}$	$i\Psi_{18}$	$-i\Psi_{19}$	$-i\Psi_{20}$	Ψ_{17}	Φ_{17}	$-\Phi_{18}$	Φ_{19}	Φ_{20}
Φ_{18}	$i\Psi_{18}$	$-i\Psi_{17}$	$-i\Psi_{20}$	$i\Psi_{19}$	Ψ_{18}	Φ_{18}	Φ_{17}	Φ_{20}	$-\Phi_{19}$
Φ_{19}	$i\Psi_{19}$	$-i\Psi_{20}$	$i\Psi_{17}$	$-i\Psi_{18}$	Ψ_{19}	Φ_{19}	Φ_{20}	$-\Phi_{17}$	Φ_{18}
Φ_{20}	$i\Psi_{20}$	$i\Psi_{19}$	$i\Psi_{18}$	$i\Psi_{17}$	Ψ_{20}	Φ_{20}	$-\Phi_{19}$	$-\Phi_{18}$	$-\Phi_{17}$

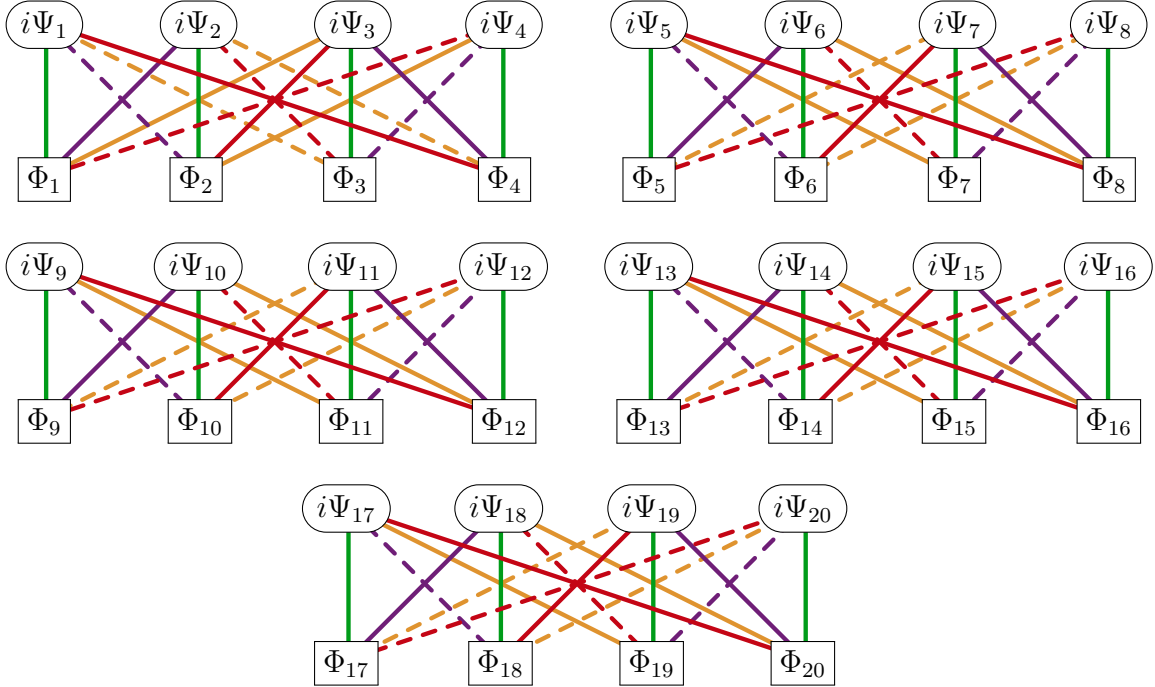


Figure 5: The $\mathfrak{m}SG$ valise adinkra. It is composed of one cis- and four trans-valise adinkras so therefore has the SUSY enantiomer numbers $n_c = 1$, $n_t = 4$. All bosons have the same engineering dimension and all fermions have the same engineering dimension.

As discussed in Section 1.1, $\mathfrak{m}SG$ is composed of cSG plus a complex linear compensating superfield, Σ . We will now show how a certain solution in our parameter space exposes the cSG adinkra as a submultiplet of the $\mathfrak{m}SG$ valise adinkra; a feature that was automatically realized for mSG in Section 2.

Consider the choice:

$$u_4 = u_5 = v_4 = v_5 = 0 \quad \Rightarrow \quad u_1 + u_2 + u_3 = v_1 + v_2 + v_3 = 1, \quad (3.18)$$

$$q_1 = q_2 = q_3 = -\frac{q_4}{N} = 1 \quad \Rightarrow \quad q_5 = n^{-1}, \quad (3.19)$$

$$p_1 = p_2 = p_3 = 1, \quad p_4 = 0 \quad \Rightarrow \quad p_5 = -3 \quad (3.20)$$

that reduces the nodal field content from Eqs. (3.16) and (3.17) to

$$\Phi = \begin{pmatrix} W_0 \\ P \\ S \\ n(h + V_0) \\ \hline u_1 A_1 + (u_2 - u_3) h_{23} \\ u_3 A_3 + (u_1 - u_2) h_{12} \\ u_2 A_2 + (u_3 - u_1) h_{31} \\ -u_1 h_{11} - u_2 h_{22} - u_3 h_{33} \\ \hline v_1 A_1 + (v_2 - v_3) h_{23} \\ v_3 A_3 + (v_1 - v_2) h_{12} \\ v_2 A_2 + (v_3 - v_1) h_{31} \\ -v_1 h_{11} - v_2 h_{22} - v_3 h_{33} \\ \hline \frac{W_0}{n} - A_0 \\ V_2 \\ -V_3 \\ -V_1 \\ \hline A_1 + 3W_1 \\ A_3 + 3W_3 \\ A_2 + 3W_2 \\ -(3n+1)h - 3nV_0 \end{pmatrix}, \quad i\Psi = \begin{pmatrix} -\beta_3 \\ -\beta_4 \\ \beta_1 \\ -\beta_2 \\ \hline -u_1 \psi_{13} + u_2 \psi_{21} + u_3 \psi_{34} \\ -u_1 \psi_{14} - u_2 \psi_{22} - u_3 \psi_{33} \\ -u_1 \psi_{11} - u_2 \psi_{23} + u_3 \psi_{32} \\ -u_1 \psi_{12} + u_2 \psi_{24} - u_3 \psi_{31} \\ \hline -v_1 \psi_{13} + v_2 \psi_{21} + v_3 \psi_{34} \\ -v_1 \psi_{14} - v_2 \psi_{22} - v_3 \psi_{33} \\ -v_1 \psi_{11} - v_2 \psi_{23} + v_3 \psi_{32} \\ -v_1 \psi_{12} + v_2 \psi_{24} - v_3 \psi_{31} \\ \hline -\frac{\beta_3}{n} - \lambda_4 - \psi_{13} + \psi_{21} + \psi_{34} \\ -\frac{\beta_4}{n} + \lambda_3 + \psi_{14} + \psi_{22} + \psi_{33} \\ -\frac{\beta_1}{n} + \lambda_2 + \psi_{11} + \psi_{23} - \psi_{32} \\ -\frac{\beta_2}{n} - \lambda_1 - \psi_{12} + \psi_{24} - \psi_{31} \\ \hline 3\beta_3 - \psi_{13} + \psi_{21} + \psi_{34} \\ -3\beta_4 - \psi_{14} - \psi_{22} - \psi_{33} \\ -3\beta_1 - \psi_{11} - \psi_{23} + \psi_{32} \\ 3\beta_2 - \psi_{12} + \psi_{24} - \psi_{31} \end{pmatrix}. \quad (3.21)$$

Under the parameter choice, Eq. (3.18), the submultiplet

$$(\Phi_5, \Phi_6, \Phi_7, \Phi_8, \Phi_9, \Phi_{10}, \Phi_{11}, \Phi_{12} | \Psi_5, \Psi_6, \Psi_7, \Psi_8, \Psi_9, \Psi_{10}, \Psi_{11}, \Psi_{12}) \quad (3.22)$$

of mSG and $\cancel{\text{m}}\text{SG}$ are *identical*. We shall identify this submultiplet with that of cSG in Section 4. The (12|12) submultiplet parameterized by the other nodes of $\cancel{\text{m}}\text{SG}$ compose an $n_c = 1$, $n_t = 2$ system, and we identify this as the complex linear compensator, per our discussion in Section 1. This is evidenced in that this submultiplet is spanned by one cis- and two trans-valise adinkras, the upper left and bottom rightmost two adinkras, respectively, in Fig. 5. Indeed, the (12|12) complex linear superfield was found in Ref. [13] to have the SUSY enantiomer numbers $n_c = 1$, $n_t = 2$.

Table 7 and the adinkra shown in Fig. 5 can be written as Eq. (1.1) with 20×20

adinkra matrices given by

$$\begin{aligned}
\mathbf{L}_1 &= \mathbf{I}_5 \otimes \mathbf{I}_4, & \mathbf{L}_2 &= i\mathbf{I}_5 \otimes \boldsymbol{\beta}_3, \\
\mathbf{L}_3 &= i \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \otimes \boldsymbol{\beta}_2, & \mathbf{L}_4 &= -i\mathbf{I}_5 \otimes \boldsymbol{\beta}_1,
\end{aligned} \tag{3.23}$$

which are written in terms of the $SO(4)$ generators in Eq. (2.25). The \mathfrak{mSG} adinkra matrices satisfy the orthogonal relationship, Eq. (2.27) and the $\mathcal{GR}(20,4)$ garden algebra

$$\begin{aligned}
\mathbf{L}_I \mathbf{R}_J + \mathbf{L}_J \mathbf{R}_I &= 2\delta_{IJ} \mathbf{I}_{20} \\
\mathbf{R}_I \mathbf{L}_J + \mathbf{R}_J \mathbf{L}_I &= 2\delta_{IJ} \mathbf{I}_{20} \quad .
\end{aligned} \tag{3.24}$$

3.3 Traces

The \mathfrak{mSG} chromocharacters are

$$\begin{aligned}
Tr[\mathbf{L}_I \mathbf{L}_J^t] &= 20 \delta_{IJ} \\
Tr[\mathbf{L}_I \mathbf{L}_J^t \mathbf{L}_K \mathbf{L}_L^t] &= 20(\delta_{IJ}\delta_{KL} - \delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}) - 12 \epsilon_{IJKL}
\end{aligned} \tag{3.25}$$

and so comparing with Eq. (2.28) we see once again that \mathfrak{mSG} has the SUSY enantiomer numbers $n_c = 1$, $n_t = 4$.

4 4D, $\mathcal{N} = 1$ Conformal Supergravity

The linearized theory of 4D, $\mathcal{N} = 1$ conformal supergravity (cSG) contains the real component fields of an axial vector *gauge* field A_μ , Majorana gravitino $\psi_{\mu a}$, and graviton $h_{\mu\nu}$. We will see in this Section that the adinkras for cSG are indeed submultiplets of both the mSG and \mathfrak{mSG} adinkras, with enantiomer numbers $n_c = 0$, $n_t = 2$.

4.1 Transformation Laws

The transformation Laws for cSG are easily found by removing the fields S and P from the mSG laws, Eq. (3.8).

$$D_a A_\mu = i(\gamma^5 \gamma^\nu)_a{}^b \partial_{[\nu} \psi_{\mu]b} - \frac{1}{2} \epsilon_\mu{}^{\nu\alpha\beta} (\gamma_\nu)_a{}^b \partial_\alpha \psi_{\beta b} \quad (4.1a)$$

$$D_a h_{\mu\nu} = \frac{1}{2} (\gamma_{(\mu})_a{}^b \psi_{\nu)b} \quad (4.1b)$$

$$D_a \psi_{\mu b} = \frac{2}{3} (\gamma^5)_{ab} A_\mu + \frac{1}{6} (\gamma^5 [\gamma_\mu, \gamma^\nu])_{ab} A_\nu - \frac{i}{2} ([\gamma^\alpha, \gamma^\beta])_{ab} \partial_\alpha h_{\beta\mu} \quad (4.1c)$$

These are a symmetry of the cSG Lagrangian

$$\begin{aligned} \mathcal{L}_{cSG} = & \frac{1}{2} h^{\mu\nu} \square^2 h_{\mu\nu} - h^{\mu\nu} \square \partial_\nu \partial_\alpha h^\alpha{}_\mu + \frac{1}{3} h^{\mu\nu} \partial_\alpha \partial_\beta \partial_\mu \partial_\nu h^{\alpha\beta} - \frac{1}{6} h \square^2 h + \\ & + \frac{1}{3} h \square \partial_\mu \partial_\nu h^{\mu\nu} - \frac{1}{6} F_{\mu\nu} F^{\mu\nu} - i \frac{1}{3} \psi_{\nu b} \partial_\mu \partial^\nu \partial^\beta (\gamma^\mu)^{bc} \psi_{\beta c} + \\ & + i \frac{1}{3} \psi_{\nu b} (\gamma^\mu)^{bc} \square \partial_\mu \psi^\nu{}_c - \frac{1}{6} \epsilon^{\nu\beta\sigma\mu} (\gamma^5 \gamma_\sigma)^{bc} \psi_{\nu b} \square \partial_\mu \psi_{\beta c} \end{aligned} \quad (4.2)$$

where the canonical $U(1)$ field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \quad (4.3)$$

With a bit of work, it can be shown that the graviton part of the cSG Lagrangian is the square of the Weyl tensor

$$\begin{aligned} C_{\alpha\mu\beta\nu} C^{\alpha\mu\beta\nu} = & \left(R_{\alpha\mu\beta\nu} - \frac{1}{2} (g_{\alpha[\beta} R_{\nu]\mu} - \eta_{\mu[\beta} R_{\nu]\alpha}) + \frac{1}{6} R g_{\alpha[\beta} g_{\nu]\mu} \right) \\ & \times \left(R^{\alpha\mu\beta\nu} - \frac{1}{2} (g^{\alpha[\beta} R^{\nu]\mu} - \eta^{\mu[\beta} R^{\nu]\alpha}) + \frac{1}{6} R g^{\alpha[\beta} g^{\nu]\mu} \right) \\ = & R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \\ = & \frac{1}{2} h^{\mu\nu} \square^2 h_{\mu\nu} - h^{\mu\nu} \square \partial_\nu \partial_\alpha h^\alpha{}_\mu + \frac{1}{3} h^{\mu\nu} \partial_\alpha \partial_\beta \partial_\mu \partial_\nu h^{\alpha\beta} + \\ & - \frac{1}{6} h \square^2 h + \frac{1}{3} h \square \partial_\mu \partial_\nu h^{\mu\nu} \end{aligned} \quad (4.4)$$

in the linear limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. This is just as is expected for conformal gravity. The following linear expansions are useful in the preceding calculation

$$R_{\alpha\mu\beta\nu} = \frac{1}{2} (\partial_\mu \partial_{[\nu} h_{\beta]\alpha} - \partial_\alpha \partial_{[\nu} h_{\beta]\mu}) \quad (4.5a)$$

$$R_{\mu\nu} = \frac{1}{2} \square h_{\mu\nu} + \frac{1}{2} \partial_\mu \partial_\nu h - \frac{1}{2} \partial_\alpha \partial_{(\nu} h^\alpha{}_{\mu)} \quad (4.5b)$$

$$R = \square h - \partial_\mu \partial_\nu h^{\mu\nu} \quad . \quad (4.5c)$$

The cSG Lagrangian possesses the linear limit conformal symmetries

$$\begin{aligned}
\delta h_{\mu\nu} &= B\eta_{\mu\nu} + \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \\
\delta A_\mu &= \partial_\mu \rho \\
\delta \psi_{\mu a} &= \partial_\mu \epsilon_a + (\gamma_\mu)_a{}^b \sigma_b \quad .
\end{aligned} \tag{4.6}$$

The cSG transformation laws satisfy the algebra

$$\begin{aligned}
\{D_a, D_b\} h_{\mu\nu} &= 2i(\gamma^\alpha)_{ab} \partial_\alpha h_{\mu\nu} - i(\gamma^\alpha)_{ab} \partial_{(\mu} h_{\nu)\alpha} \\
\{D_a, D_b\} A_\mu &= 2i(\gamma^\nu)_{ab} \partial_\nu A_\mu \\
\{D_a, D_b\} \psi_{\mu c} &= 2i(\gamma^\alpha)_{ab} \partial_\alpha \psi_{\mu c} - i\partial_\mu \varphi_{abc} - i(\gamma_\mu)_c{}^d \sigma_{abd}
\end{aligned} \tag{4.7}$$

where φ_{abc} is as before in both the minimal and non-minimal representations, Eq. (2.6), and another piece has arisen on the right hand side of the gravitino algebra

$$\sigma_{abd} = \frac{1}{3} \left((\gamma^{[\alpha})_{ab} (\gamma^{\beta]})_d{}^e + i\epsilon^{\nu\rho\alpha\beta} (\gamma_\nu)_{ab} (\gamma^5 \gamma_\rho)_d{}^e \right) \partial_\alpha \psi_{\beta e} \quad . \tag{4.8}$$

The term involving σ_{abd} is proportional to $(\gamma_\mu)_c{}^d$ and so is a consequence of the related symmetry of the cSG Lagrangian depicted in the last line of Eq. (4.6). The auxiliary fields in both the minimal and non-minimal cases, serve to remove this term, which necessarily reduces the full conformal symmetry group to Poincaré.

4.2 One-Dimensional Reduction

We use temporal gauge, Eq. (2.9), as before for $h_{\mu\nu}$ and $\psi_{\mu a}$, but now also for the axial $U(1)$ vector

$$A_0 = 0 \quad . \tag{4.9}$$

In this gauge, the Lagrangian (4.2) reduced to the 0 brane becomes

$$\begin{aligned}
\mathcal{L}_{cSG}^{(0)} &= \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \ddot{h}_{ij} \ddot{h}_{ij} - \frac{1}{6} \ddot{h}^2 - \frac{1}{3} \sum_{i=1}^3 \dot{A}_i \dot{A}_i + \frac{i}{3} \sum_{i=1}^3 \sum_{b=1}^4 \dot{\psi}_{ib} \ddot{\psi}_{ib} + \\
&\quad - \frac{i}{3} \left(\dot{\psi}_{11} \ddot{\psi}_{23} - \dot{\psi}_{13} \ddot{\psi}_{21} - \dot{\psi}_{12} \ddot{\psi}_{24} + \dot{\psi}_{14} \ddot{\psi}_{22} \right) + \\
&\quad - \frac{i}{3} \left(\dot{\psi}_{31} \ddot{\psi}_{12} - \dot{\psi}_{32} \ddot{\psi}_{11} + \dot{\psi}_{33} \ddot{\psi}_{14} - \dot{\psi}_{34} \ddot{\psi}_{13} \right) + \\
&\quad - \frac{i}{3} \left(\dot{\psi}_{21} \ddot{\psi}_{34} - \dot{\psi}_{24} \ddot{\psi}_{31} + \dot{\psi}_{22} \ddot{\psi}_{33} - \dot{\psi}_{23} \ddot{\psi}_{32} \right) \quad .
\end{aligned} \tag{4.10}$$

Its symmetries, Eqs. (4.6), become

$$\begin{aligned}
\delta h_{ij} &= B \delta_{ij} \quad , \quad \delta A_i = 0 \quad , \quad i, j = 1, 2, 3, \\
\delta \psi_{11} &= \sigma_2 \quad , \quad \delta \psi_{12} = \sigma_1 \quad , \quad \delta \psi_{13} = \sigma_4 \quad , \quad \delta \psi_{14} = \sigma_3 \quad , \\
\delta \psi_{21} &= -\sigma_4 \quad , \quad \delta \psi_{22} = \sigma_3 \quad , \quad \delta \psi_{23} = \sigma_2 \quad , \quad \delta \psi_{24} = -\sigma_1 \quad , \\
\delta \psi_{31} &= \sigma_1 \quad , \quad \delta \psi_{32} = -\sigma_2 \quad , \quad \delta \psi_{33} = \sigma_3 \quad , \quad \delta \psi_{34} = -\sigma_4 \quad ,
\end{aligned} \tag{4.11}$$

with the constraints

$$\begin{aligned}
B &= 2\dot{\Lambda}_0 \quad , \quad \Lambda_i = \text{constant} \quad , \quad \rho = \text{constant} \\
\dot{\epsilon}_1 &= -\sigma_2 \quad , \quad \dot{\epsilon}_2 = \sigma_1 \quad , \quad \dot{\epsilon}_3 = \sigma_4 \quad , \quad \dot{\epsilon}_4 = -\sigma_3 \quad ,
\end{aligned} \tag{4.12}$$

that serve to maintain temporal gauge, Eqs. (2.9) and (4.9).

The adinkranization proceeds precisely as in Section 2, but now with $S = P = A_0 = 0$. From the solutions, Eqs. 2.19 and 2.20, we see there is *no* way to decompose cSG into a cis-valise adinkra, though there is the same degeneracy in decomposing into trans-valises as there was for mSG. Since cSG has $k = 2 = n_c + n_t$, we conclude that cSG has SUSY enantiomer numbers

$$n_c = 0 \quad , \quad n_t = 2 \quad . \tag{4.13}$$

The node definitions for the cSG adinkra, Fig. 6, are as in Eqs. (4.14), from which the 0-brane reduced transformation laws in Table 8 can be read. The parameters in Eqs. (4.14) are constrained as in Eq. (2.23).

$$\Phi = \begin{pmatrix} u_1 A_1 + (u_2 - u_3) \dot{h}_{23} \\ u_3 A_3 + (u_1 - u_2) \dot{h}_{12} \\ u_2 A_2 + (u_3 - u_1) \dot{h}_{31} \\ \frac{-u_1 \dot{h}_{11} - u_2 \dot{h}_{22} - u_3 \dot{h}_{33}}{v_1 A_1 + (v_2 - v_3) \dot{h}_{23}} \\ v_3 A_3 + (v_1 - v_2) \dot{h}_{12} \\ v_2 A_2 + (v_3 - v_1) \dot{h}_{31} \\ -v_1 \dot{h}_{11} - v_2 \dot{h}_{22} - v_3 \dot{h}_{33} \end{pmatrix} \quad , \quad i\Psi = \begin{pmatrix} -u_1 \dot{\psi}_{13} + u_2 \dot{\psi}_{21} + u_3 \dot{\psi}_{34} \\ -u_1 \dot{\psi}_{14} - u_2 \dot{\psi}_{22} - u_3 \dot{\psi}_{33} \\ -u_1 \dot{\psi}_{11} - u_2 \dot{\psi}_{23} + u_3 \dot{\psi}_{32} \\ \frac{-u_1 \dot{\psi}_{12} + u_2 \dot{\psi}_{24} - u_3 \dot{\psi}_{31}}{-v_1 \dot{\psi}_{13} + v_2 \dot{\psi}_{21} + v_3 \dot{\psi}_{34}} \\ -v_1 \dot{\psi}_{14} - v_2 \dot{\psi}_{22} - v_3 \dot{\psi}_{33} \\ -v_1 \dot{\psi}_{11} - v_2 \dot{\psi}_{23} + v_3 \dot{\psi}_{32} \\ -v_1 \dot{\psi}_{12} + v_2 \dot{\psi}_{24} - v_3 \dot{\psi}_{31} \end{pmatrix} \tag{4.14}$$

These node definitions have all the 0-brane symmetries of the cSG Lagrangian, Eq. (4.11). Note that the nodes in the cSG valise adinkra are precisely the sub-multiplet, (3.22), that showed up in the mSG and the $\overline{\text{mSG}}$ under the parameter choice Eq. (3.18).

From the node definitions in Eqs. 4.14, it would appear at first glance that there are more degrees of freedom in the original fields than the number of nodes. This is not the case precisely because the nodes are invariant with respect to the 0-brane

Table 8: Zero-brane reduced cSG transformation rules in the adinkraic representation in Eqs. (4.14).

	D ₁	D ₂	D ₃	D ₄		D ₁	D ₂	D ₃	D ₄
Φ ₁	$i\Psi_1$	$i\Psi_2$	$-i\Psi_3$	$-i\Psi_4$	Ψ ₁	Φ ₁	$-\Phi_2$	Φ ₃	Φ ₄
Φ ₂	$i\Psi_2$	$-i\Psi_1$	$-i\Psi_4$	$i\Psi_3$	Ψ ₂	Φ ₂	Φ ₁	Φ ₄	$-\Phi_3$
Φ ₃	$i\Psi_3$	$-i\Psi_4$	$i\Psi_1$	$-i\Psi_2$	Ψ ₃	Φ ₃	Φ ₄	$-\Phi_1$	Φ ₂
Φ ₄	$i\Psi_4$	$i\Psi_3$	$i\Psi_2$	$i\Psi_1$	Ψ ₄	Φ ₄	$-\Phi_3$	$-\Phi_2$	$-\Phi_1$
Φ ₅	$i\Psi_5$	$i\Psi_6$	$-i\Psi_7$	$-i\Psi_8$	Ψ ₅	Φ ₅	$-\Phi_6$	Φ ₇	Φ ₈
Φ ₆	$i\Psi_6$	$-i\Psi_5$	$-i\Psi_8$	$i\Psi_7$	Ψ ₆	Φ ₆	Φ ₅	Φ ₈	$-\Phi_7$
Φ ₇	$i\Psi_7$	$-i\Psi_8$	$i\Psi_5$	$-i\Psi_6$	Ψ ₇	Φ ₇	Φ ₈	$-\Phi_5$	Φ ₆
Φ ₈	$i\Psi_8$	$i\Psi_7$	$i\Psi_6$	$i\Psi_5$	Ψ ₈	Φ ₈	$-\Phi_7$	$-\Phi_6$	$-\Phi_5$

symmetries, Eq. (4.11). These residual symmetries remove one more degree of freedom from the graviton, leaving it with a total of five, and four more from the gravitino, leaving it with eight. Adding the three degrees of freedom from the completely gauge fixed A_i field brings the degrees of freedom to (8|8) once all gauge degrees of freedom encoded in Eq. (4.6) have been removed.

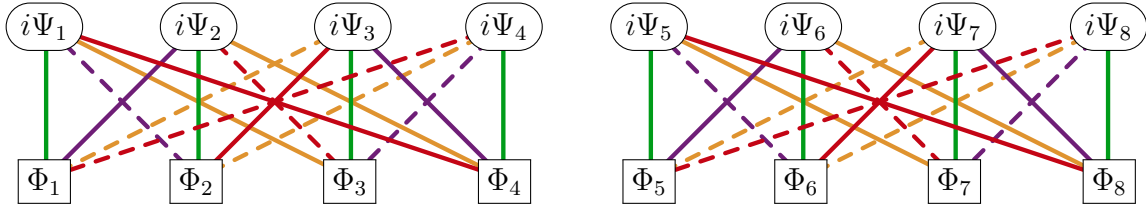


Figure 6: The cSG valise adinkra. It has the exact same nodal content as the two mSG trans-adinkra pieces in Fig. 3 and two of the four ϕSG trans-adinkra pieces in Fig. 5 under the parameter choice (3.18). The cSG SUSY enantiomer numbers are therefore $n_c = 0$, $n_t = 2$. The engineering dimensions of all bosons are the same and the engineering dimensions of all fermions are the same.

The cSG adinkra matrices can be read off either Table 8 or Fig. 6

$$\begin{aligned} \mathbf{L}_1 &= \mathbf{I}_2 \otimes \mathbf{I}_4 & , & \quad \mathbf{L}_2 = i\mathbf{I}_2 \otimes \boldsymbol{\beta}_3 & , \\ \mathbf{L}_3 &= -i\mathbf{I}_2 \otimes \boldsymbol{\beta}_2 & , & \quad \mathbf{L}_4 = -i\mathbf{I}_2 \otimes \boldsymbol{\beta}_1 & . \end{aligned} \quad (4.15)$$

These satisfy the orthogonality relationship, Eq. (2.27), and the $\mathcal{GR}(8, 4)$ algebra

$$\begin{aligned} \mathbf{L}_I \mathbf{R}_J + \mathbf{L}_J \mathbf{R}_I &= 2\delta_{IJ} \mathbf{I}_8 \\ \mathbf{R}_I \mathbf{L}_J + \mathbf{R}_J \mathbf{L}_I &= 2\delta_{IJ} \mathbf{I}_8 \quad . \end{aligned} \quad (4.16)$$

4.3 Traces

The chromocharacters for cSG are

$$\begin{aligned} Tr[\mathbf{L}_I \mathbf{L}_J^t] &= 8 \delta_{IJ} \\ Tr[\mathbf{L}_I \mathbf{L}_J^t \mathbf{L}_K \mathbf{L}_L^t] &= 8(\delta_{IJ}\delta_{KL} - \delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}) - 8 \epsilon_{IJKL} \quad . \end{aligned} \quad (4.17)$$

Comparing with our master formula Eq. (2.28), once again we see that the cSG SUSY enantiomer numbers are $n_c = 0$, $n_t = 2$.

4.4 The View From The Conformal Perspective

The collection of fields in (4.10) and (4.11) together with their associated L-matrices in (4.15) along with their associated adinkras in Fig. 6 are highly significant in another way. One can easily see upon comparison between the bosonic and fermionic fields in (4.14), the bottom eight lines of the bosons and fermions in (2.22), and lines five through 12 of the bosons and fermions in (3.21) that these all correspond to the same structure.

Let us rename these structures according to the definitions

$$\mathcal{H}_{i\Phi}^{(V)} = \begin{pmatrix} u_1 A_1 + (u_2 - u_3) \dot{h}_{23} \\ u_3 A_3 + (u_1 - u_2) \dot{h}_{12} \\ u_2 A_2 + (u_3 - u_1) \dot{h}_{31} \\ \frac{-u_1 \dot{h}_{11} - u_2 \dot{h}_{22} - u_3 \dot{h}_{33}}{v_1 A_1 + (v_2 - v_3) \dot{h}_{23}} \\ v_3 A_3 + (v_1 - v_2) \dot{h}_{12} \\ v_2 A_2 + (v_3 - v_1) \dot{h}_{31} \\ -v_1 \dot{h}_{11} - v_2 \dot{h}_{22} - v_3 \dot{h}_{33} \end{pmatrix}, \quad i\mathcal{H}_{\hat{j}\Psi}^{(V)} = \begin{pmatrix} -u_1 \dot{\psi}_{13} + u_2 \dot{\psi}_{21} + u_3 \dot{\psi}_{34} \\ -u_1 \dot{\psi}_{14} - u_2 \dot{\psi}_{22} - u_3 \dot{\psi}_{33} \\ -u_1 \dot{\psi}_{11} - u_2 \dot{\psi}_{23} + u_3 \dot{\psi}_{32} \\ \frac{-u_1 \dot{\psi}_{12} + u_2 \dot{\psi}_{24} - u_3 \dot{\psi}_{31}}{-v_1 \dot{\psi}_{13} + v_2 \dot{\psi}_{21} + v_3 \dot{\psi}_{34}} \\ -v_1 \dot{\psi}_{14} - v_2 \dot{\psi}_{22} - v_3 \dot{\psi}_{33} \\ -v_1 \dot{\psi}_{11} - v_2 \dot{\psi}_{23} + v_3 \dot{\psi}_{32} \\ -v_1 \dot{\psi}_{12} + v_2 \dot{\psi}_{24} - v_3 \dot{\psi}_{31} \end{pmatrix} \quad (4.18)$$

and re-express the supersymmetrical D-algebra in the form of two equations

$$D_I \mathcal{H}_{i\Phi}^{(V)} = i(L_I)_{ij} \mathcal{H}_{\hat{j}\Psi}^{(V)}, \quad D_I \mathcal{H}_{\hat{j}\Psi}^{(V)} = (R_I)_{\hat{j}i} \frac{d}{dt} \mathcal{H}_{i\Phi}^{(V)} \quad (4.19)$$

where the L-matrices and corresponding R-matrices are defined by Eq. (4.15). This particular valise describes 4D, $\mathcal{N} = 1$ conformal supergravity. The fact that it universally occurs in each of the supergravity formulation is equivalent to two well known facts in other approaches:

- a. In the superspace approach of [8] there was presented a description of supergravity in terms of unconstrained ‘prepotential’ superfields. One of the features of the construction was to show that in fact, all off-shell version of SG theory when written in terms of unconstrained superfields can be split into a ‘conformal prepotential superfield’ and a ‘conformal compensator superfield.’ Although the former is unique, the latter was shown not to be so. This possibility of different choices for the conformal compensator accounts for the different auxiliary field structures that can occur in off-shell descriptions. This work was also the first to show that even in the confines of a Poincaré theory, there is an important role for the symmetries of a conformal theory.

- b. In the conformal component-level approach of [17] which began after the work of Ref. [8], once more one sees that there is a sub-multiplet in all off-shell supergravity theories that consists solely of the fields required to describe conformal supergravity. In this approach the component fields that appear over and above these arise (as they do in the superfield approach) as the result of the breaking of conformal symmetry.

The task of understanding how the spacetime superconformal group is embedded with the approach of an adinkra-based formulation is an important task to be carried out in future research along these lines.

5 Synthesis

In this section we synthesize the main results of the paper into a cohesive framework, including relations to the previous two Refs. [12, 13]. Consider the supercharacter [18]

$$\chi_\rho(t, a, b) = \text{Tr}_\rho \left[(-1)^F t^{2\Delta} e^{iaM_{01}} e^{ibM_{23}} \right]. \quad (5.1)$$

Here F is the fermion number, M_{01} and M_{23} are Lorentz generators, and Δ computes the engineering dimension of the field in the representation ρ normalized so that a physical spinor has dimension $\Delta = \frac{3}{2}$. In all cases of interest to us, Δ is integral iff $F = 0$ so that we may combine the first two terms to give

$$\chi_\rho(t, a, b) = \text{Tr}_\rho \left[(-t)^{2\Delta} e^{iaM_{01}} e^{ibM_{23}} \right]. \quad (5.2)$$

If we are uninterested in the dimension of the fields in the representation ρ and if, furthermore, we forget about the grading by fermion number, we may set $t = -1$ and study the resulting $Spin(3, 1)$ character instead. This quantity is related by Wick rotation to the chromocharacters computed previously.⁹ That is, we consider instead of χ , a “twisted” version $\tilde{\chi}$ on $Spin(4)_R$

$$\tilde{\chi}_\rho = \text{Tr}_\rho \left[(-t)^{2\Delta} e^{iaM_{12}} e^{ibM_{34}} \right] \quad (5.3)$$

$$(M_{IJ})_i{}^j = \frac{i}{4} \left[(L_I)_i{}^{\hat{k}} (R_J)_{\hat{k}}{}^j - (L_J)_i{}^{\hat{k}} (R_I)_{\hat{k}}{}^j \right] \quad (5.4)$$

⁹An advantage of this compactification is that the representations are replaced with finite-dimensional analogues which possess no gauge freedom.

Notice that in our definition of ‘twisted,’ the usual Lorentz generators are replaced by the matrices defined in Eq. (5.4) that are not Lorentz generators, but we will use them as if they are. Then, for example, any garden algebra satisfying the orthogonality relation (2.27) and the chromocharacter formula (2.28) will have

$$\begin{aligned} \left. \frac{\partial}{\partial a} \frac{\partial}{\partial b} \tilde{\chi} \right|_{a,b=0} &= 4(n_c - n_t) \\ \left(\frac{\partial}{\partial a} \right)^2 \tilde{\chi} \Big|_{a,b=0} &= \left(\frac{\partial}{\partial b} \right)^2 \tilde{\chi} \Big|_{a,b=0} = -4(n_c + n_t) \end{aligned} \quad (5.5)$$

and we recover the enantiomer numbers. In this way, “adinkranization” may be thought of as an algorithm for computing the characters of the (Wick-rotated) super-Poincaré group.

More explicitly, the calculation of the twisted character for the cis, trans, real-unconstrained (\mathbb{R}), cSG, mSG and complex linear superfield (Σ), and \mathfrak{m} SG are

$$\begin{aligned} \tilde{\chi}_{\text{cis}}(a, b) &= 4 \cos\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) + 4 \sin\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \\ \tilde{\chi}_{\text{trans}}(a, b) &= 4 \cos\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) - 4 \sin\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \\ \tilde{\chi}_{\mathbb{R}}(a, b) &= 8 \cos\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) \\ \tilde{\chi}_{\text{cSG}}(a, b) &= 8 \cos\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) - 8 \sin\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \\ \tilde{\chi}_{\text{mSG}}(a, b) = \tilde{\chi}_{\Sigma}(a, b) &= 12 \cos\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) - 4 \sin\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \\ \tilde{\chi}_{\mathfrak{m}\text{SG}}(a, b) &= 20 \cos\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) - 12 \sin\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right). \end{aligned} \quad (5.6)$$

In fact, these representations all fit into the formula:

$$\tilde{\chi}(a, b) = 4(n_c + n_t) \cos\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) + 4(n_c - n_t) \sin\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \quad . \quad (5.7)$$

By taking partial derivatives, we clearly find the correct (n_c, n_t) numbers. By way of comparison, the characters for the 2-component (anti-)Weyl, the analogue of the (cis) trans representation, and 4-component Dirac spinor representations of $Spin(3, 1)$ are

$$\begin{aligned}
\chi_{\text{Weyl}}(a, b) &= 2 \cosh\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) + 2i \sinh\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \\
\chi_{\overline{\text{Weyl}}}(a, b) &= 2 \cosh\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) - 2i \sinh\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) \\
\chi_{\text{Dirac}}(a, b) &= 4 \cosh\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right).
\end{aligned} \tag{5.8}$$

Noting from Ref. [12] that the chiral multiplet is in the cis-representation, we see that the characters (5.6) obey the superfield equation

$$\text{complex unconstrained} = \text{chiral} + \text{complex linear}. \tag{5.9}$$

Similarly, we comment that the results for the gauge-fixed vector multiplet and tensor multiplet given in the literature are consistent with their identification as real-linear multiplets. For example, the characters (5.6) obey the superfield equation

$$\begin{aligned}
\text{gauge-fixed vector} &= \text{real unconstrained} - (\text{chiral} + \overline{\text{chiral}}) \\
&= \text{real-linear}.
\end{aligned} \tag{5.10}$$

It is important to note that in this equation $(\text{chiral} + \overline{\text{chiral}})$ is the real part of the chiral superfield, which can be identified with *one* cis-representation in Eq. (5.6) as these are all real representations.

We represent all of this in the following table:

	$\chi_{\text{ral}} (4 4)$	$\mathbb{R}\text{-lin.} (4 4)$	$\mathbb{R} (8 8)$	$\mathbb{C}\text{-lin.} (12 12)$	$\mathbb{C} (16 16)$	
$n_c + n_t$	1	1	2	3	4	(5.11)
$n_c - n_t$	1	-1	0	-1	0	

The super-dimensions of these representations are indicated as $(d_b|d_f)$.

Just as there are two inequivalent $(4|4)$ -dimensional representations (chiral and real-linear), there is a second $(8|8)$ -dimensional one: conformal supergravity. Together with the equations

$$\begin{aligned}
\text{minimal SG} &= \text{conformal SG} + \text{chiral compensator} \\
\text{non-minimal SG} &= \text{conformal SG} + \text{complex-linear compensator}
\end{aligned} \tag{5.12}$$

we obtain the following table:

	cSG (8 8)	mSG (12 12)	ϖSG (20 20)	
$n_c + n_t$	2	3	5	(5.13)
$n_c - n_t$	-2	-1	-3	

This completes the list of all known off-shell $4D$, $\mathcal{N} = 1$ representations whose adinkras have been reported to date [12, 13].

6 Conclusion

In this paper, the SUSY enantiomer numbers and adinkras for mSG, ϖSG, and cSG were explicitly derived. It was found that these numbers indeed exhibit additive behavior among the different representations, like characters in group theory. The base multiplet for supergravity is cSG, with SUSY enantiomer numbers $(n_c, n_t) = (0, 2)$. The mSG multiplet is cSG plus a chiral compensator superfield with SUSY enantiomer numbers $(n_c, n_t) = (1, 0)$. We indeed found the enantiomer numbers for mSG to be the sum of the enantiomer numbers for cSG and the chiral superfield: $(n_c, n_t) = (0, 2) + (1, 0) = (1, 2)$. The same holds true for ϖSG, which is cSG added to a complex linear compensator superfield, and all other multiplets investigated so far [12, 13]. This leads us to believe that SUSY enantiomer numbers are indeed characters describing the superfield content, and the adinkras are pictorial representations of these characters.

We also unveiled a simple procedure for finding SUSY enantiomer numbers. This utilizes the cis- and trans-valise adinkra pictures and the 0-brane transformation laws for the multiplet whose enantiomer numbers are sought. The procedure is to force the 0-brane transformation laws to fit into either the cis- or trans-valise adinkra, leading to constraints on the possible linear combinations of the fields in the multiplet that define the nodes in the irreducible adinkras.

For the adinkranization procedure to produce unambiguous SUSY enantiomer numbers, it appears a $4D$, $\mathcal{N} = 1$ off-shell multiplet described by a $\mathcal{GR}(d, N)$ algebra can have only one of four sets of SUSY enantiomer numbers:

$$n_c = 0 \quad , \quad n_t = \frac{d}{4} \quad , \quad (6.1a)$$

$$n_c = 1 \quad , \quad n_t = \frac{d}{4} - 1 \quad , \quad (6.1b)$$

$$n_t = 0 \quad , \quad n_c = \frac{d}{4} \quad , \quad (6.1c)$$

$$n_t = 1 \quad , \quad n_c = \frac{d}{4} - 1 \quad . \quad (6.1d)$$

In this paper, we showed that cSG satisfies case (6.1a) and mSG and \cancel{m} SG both satisfy case (6.1b). The three 4D, $\mathcal{N} = 1$ off-shell cases investigated in Ref. [12], the chiral, vector, and tensor multiplets satisfy cases (6.1c), (6.1a), and (6.1a), respectively. These three have trivial solution spaces, i.e., there is no parameter space of solutions for adinkranization as was found for the mSG, \cancel{m} SG, and cSG. The 4D, $\mathcal{N} = 1$ off-shell cases investigated in Ref. [13], the real scalar superfield and complex linear superfield, both satisfy case (6.1b), though the real scalar superfield simultaneously satisfies case (6.1d) since $d = 8$. The solution space for the real scalar superfield is trivial as was the case for the tensor, vector, and chiral multiplets. The complex linear superfield on the other hand has a non-trivial solution space, much like that of mSG. This was not explicitly derived in Ref. [13], though a quick analysis of these multiplets with the procedure developed in this paper shows that it is indeed true.

Ongoing work that will be part of a future publication has already produced case (6.1d) for the $d = 20$, 4D, $\mathcal{N} = 1$ gravitino-matter multiplet. The real scalar superfield is the only other known representation which fits into this case, though it does so with a trivial solution space. The gravitino-matter multiplet is therefore the only known instance of case (6.1d) for which the solution space is non-trivial and is in terms of the cis-enantiomer number rather than the trans-enantiomer number. The roles of cis and trans between the gravitino-matter multiplet and all other known non-trivial multiplets are reversed!

Moving forward, we plan on continuing to investigate adinkranization of higher superspin off-shell systems. For these higher d systems either the cis or the trans numbers will increase (whichever increases, the other will be held at zero or one) if our adinkranization conjecture (6.1) is to hold. Now the questions seem to be, is it the cis or the trans number that will increase for these higher d systems, what is the pattern, and what information are the cis and trans numbers storing?

“Our nation must come together to unite.” - George W. Bush

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A Definitions and Conventions

We will use the real representation of the γ matrices as in Refs. [12, 13]

$$\begin{aligned} (\gamma^0)_a^b &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & (\gamma^1)_a^b &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ (\gamma^2)_a^b &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & (\gamma^3)_a^b &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned} \quad (\text{A.1})$$

$$(\gamma^5)_a^b \equiv i(\gamma^0\gamma^1\gamma^2\gamma^3)_a^b = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.2})$$

We also use the following conventions for the totally antisymmetric Levi-Civita tensor and the $SO(1,3)$ generators for spinors

$$\begin{aligned} \epsilon_{0123} &= -\epsilon^{0123} = 1 \quad \text{and totally anti-symmetric,} \\ \sigma^{\mu\nu} &\equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \end{aligned} \quad (\text{A.3})$$

Einstein summation convention is assumed throughout, for example

$$A^\mu A_\mu \equiv \sum_{\mu=0}^3 A^\mu A_\mu = A^0 A_0 + A^1 A_1 + A^2 A_2 + A^3 A_3 \quad . \quad (\text{A.4})$$

All lower case Greek indices $\mu, \nu, \alpha, \beta, \dots$ are space-time indices and are raised and lowered with the Minkowski metric

$$\eta_{\mu\nu} = \eta_{\nu\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \eta^{\mu\nu} = \eta^{\nu\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad (\text{A.5})$$

as

$$A_\mu = \eta_{\mu\nu} A^\nu \quad , \quad A^\mu = \eta^{\mu\nu} A_\nu \quad . \quad (\text{A.6})$$

Lower case Latin a, b, c, \dots are fermionic indices and are raised and lowered with the spinor metric

$$C_{ab} = -C_{ba} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad , \quad C^{ab} = -C^{ba} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (\text{A.7})$$

according to the *northwest-southeast* rules:

$$\psi^a = C^{ab} \psi_b \quad , \quad \psi_a = \psi^b C_{ba} \quad . \quad (\text{A.8})$$

Symmetrization and anti-symmetrization are defined as follows without any normalization:

$$(\gamma_{(\mu})_a{}^b \psi_{\nu)b} \equiv (\gamma_\mu)_a{}^b \psi_{\nu b} + (\gamma_\nu)_a{}^b \psi_{\mu b} \quad , \quad (\gamma_{[\mu})_a{}^b \psi_{\nu]b} \equiv (\gamma_\mu)_a{}^b \psi_{\nu b} - (\gamma_\nu)_a{}^b \psi_{\mu b} \quad . \quad (\text{A.9})$$

We use the following conventions for the Riemann and Ricci tensors, Ricci scalar, and Christoffel symbols:

$$R^\alpha{}_{\mu\beta\nu} = \partial_\nu \Gamma^\alpha{}_{\mu\beta} - \partial_\beta \Gamma^\alpha{}_{\mu\nu} + \Gamma^\rho{}_{\mu\beta} \Gamma^\alpha{}_{\nu\rho} - \Gamma^\rho{}_{\mu\nu} \Gamma^\alpha{}_{\beta\rho} \quad (\text{A.10a})$$

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \partial_\nu \Gamma^\alpha{}_{\mu\alpha} - \partial_\alpha \Gamma^\alpha{}_{\mu\nu} + \Gamma^\alpha{}_{\mu\beta} \Gamma^\beta{}_{\nu\alpha} - \Gamma^\alpha{}_{\mu\nu} \Gamma^\beta{}_{\alpha\beta} \quad (\text{A.10b})$$

$$R = g^{\mu\nu} R_{\mu\nu} \quad (\text{A.10c})$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\partial_{(\beta}g_{\alpha)\nu} - \partial_\nu g_{\alpha\beta}) \quad . \quad (\text{A.10d})$$

We linearize with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad (\text{A.11})$$

after which indices are raised and lowered by the Minkowski metric $\eta_{\mu\nu}$ and $h_{\mu\nu}$ is referred to as the graviton. For instance, we have the linearized Christoffel symbol

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(\partial_{(\beta}h_{\alpha)\nu} - \partial_\nu h_{\alpha\beta}) \quad . \quad (\text{A.12})$$

We define the d'Alembertian operator as

$$\square \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu \quad (\text{A.13})$$

and h denotes the trace of the graviton which is symmetric

$$h \equiv \eta^{\mu\nu}h_{\mu\nu} \quad , \quad h_{\mu\nu} = h_{\nu\mu} \quad . \quad (\text{A.14})$$

Conventions for reducing $4D$ transformation laws to the 0-brane as well as all conventions for drawing adinkras from these transformation laws are as in Parts I and II [12, 13]. A quick review of these rules can be found in Appendix A of Part II [13].

B Proof of Most General mSG Lagrangian, Transformation Laws, and Algebra in Majorana Components

The most general linear $4D$, $\mathcal{N} = 1$ minimal SUGRA Lagrangian with component fields in a real Majorana representation as used in this paper takes the form

$$\begin{aligned} \mathcal{L} = & h_0 \left(-\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \frac{1}{2}\partial^\alpha h\partial_\alpha h - \partial^\alpha h\partial^\beta h_{\alpha\beta} + \partial^\mu h_{\mu\nu}\partial_\alpha h^{\alpha\nu} \right) + \\ & -s_0\frac{1}{2}S^2 - p_0\frac{1}{2}P^2 + a_0\frac{1}{2}A_\mu A^\mu - f_0\frac{1}{2}\psi_{\mu a}\epsilon^{\mu\nu\alpha\beta}(\gamma^5\gamma_\nu)^{ab}\partial_\alpha\psi_{\beta b} \end{aligned} \quad (\text{B.1})$$

where a_0 , s_0 , p_0 , h_0 , and f_0 are constants to be determined via supersymmetry. Variation of the Lagrangian with respect to $h_{\mu\nu}$ results in

$$\delta\mathcal{L} = -2\delta h^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \right) \quad (\text{B.2})$$

where $R_{\mu\nu}$ is the linearized Ricci tensor

$$\begin{aligned} R_{\mu\nu} &= -\partial_\alpha \Gamma_{\mu\nu}^\alpha + \partial_\nu \Gamma_{\mu\alpha}^\alpha - \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\beta}^\beta + \Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta \\ &= -\frac{1}{2} \partial_\beta \partial_{(\mu} h_{\nu)}^\beta + \frac{1}{2} \partial_\alpha \partial^\alpha h_{\mu\nu} + \frac{1}{2} \partial_\mu \partial_\nu h \end{aligned} \quad (\text{B.3})$$

where we have the linearized Christoffel symbol

$$\begin{aligned} \Gamma_{\alpha\beta}^\mu &= \frac{1}{2} (\eta^{\mu\nu} - h^{\mu\nu}) (\partial_{(\beta} (\eta_{\alpha)\nu} + h_{\alpha)\nu}) - \partial_\nu (\eta_{\alpha\beta} + h_{\alpha\beta}) \\ &= \frac{1}{2} \eta^{\mu\nu} (\partial_{(\beta} h_{\alpha)\nu} - \partial_\nu h_{\alpha\beta}) \quad . \end{aligned} \quad (\text{B.4})$$

This confirms we are using the correct linearized supergravity Lagrangian as it produces Einstein's equations in vacuum for the linear theory.

In the following subsections, we will first impose this supersymmetry via a set of general transformation laws, then use closure of the algebra of these transformation laws to find a final solution for these constants. This will leave us, up to field redefinition symmetries present in the Lagrangian, the most general Lagrangian and set of supersymmetric transformation laws.

B.1 Transformation Laws

The most general transformation laws that are an invariant of the Lagrangian take the form

$$D_a S = i s_1 (\sigma^{\mu\nu})_a^b \partial_\mu \psi_{\nu b} \quad (\text{B.5a})$$

$$D_a P = p_1 (\gamma^5 \sigma^{\mu\nu})_a^b \partial_\mu \psi_{\nu b} \quad (\text{B.5b})$$

$$D_a A_\mu = i a_1 (\gamma^5 \gamma^\nu)_a^b \partial_{[\nu} \psi_{\mu]b} + a_2 \epsilon_\mu^{\nu\alpha\beta} (\gamma_\nu)_a^b \partial_\alpha \psi_{\beta b} \quad (\text{B.5c})$$

$$D_a h_{\mu\nu} = h_1 (\gamma_\mu)_a^b \psi_{\nu b} + h_1 (\gamma_\nu)_a^b \psi_{\mu b} + h_2 \eta_{\mu\nu} (\gamma^\alpha)_a^b \psi_{\alpha b} \quad (\text{B.5d})$$

$$\begin{aligned} D_a \psi_{\mu b} &= i f_1 (\gamma_\mu)_{ab} S + f_2 (\gamma^5 \gamma_\mu)_{ab} P + f_3 (\gamma^5)_{ab} A_\mu + i f_4 (\gamma^5 \sigma_\mu^\nu)_{ab} A_\nu + \\ &\quad + f_5 (\sigma^{\alpha\beta})_{ab} \partial_\alpha h_{\beta\mu} + i f_6 C_{ab} \partial_\alpha h_\mu^\alpha + i f_7 C_{ab} \partial_\mu h + f_8 (\sigma_\mu^\nu)_{ab} \partial_\nu h + \\ &\quad + f_9 (\sigma_\mu^\alpha)_{ab} \partial_\beta h_\alpha^\beta \quad . \end{aligned} \quad (\text{B.5e})$$

A few comments about the things that have led us to this present form. We have excluded all terms on the right hand sides of the transformation laws for S , P , and A_μ that do not obey the symmetry in Eq. (2.8). It is also notable that the spin connection

shows up in the f_5 term in the transformation laws for $\psi_{\mu b}$, and is equivalent to

$$\omega_{\mu\alpha\beta} = g_{\alpha\nu} e_{\beta}^{\rho} \partial_{\mu} e_{\rho}^{\nu} + g_{\alpha\nu} \Gamma_{\mu\beta}^{\nu} = \frac{1}{2} \partial_{[\beta} h_{\alpha]\mu} \quad (\text{B.6})$$

where the linear frame fields are defined as

$$e_{\mu}^{\underline{\mu}} \equiv \delta_{\mu}^{\underline{\mu}} + \frac{1}{2} f_{\mu}^{\underline{\mu}} \quad (\text{B.7})$$

$$e_{\underline{\mu}}^{\mu} \equiv \delta_{\underline{\mu}}^{\mu} - \frac{1}{2} f_{\underline{\mu}}^{\mu} \quad (\text{B.8})$$

and defined to satisfy

$$h_{\mu\nu} \equiv \eta_{\underline{\mu}\underline{\alpha}} \delta_{\nu}^{\underline{\alpha}} f_{\mu}^{\underline{\mu}} = \eta_{\nu\alpha} \delta_{\mu}^{\alpha} f_{\nu}^{\underline{\mu}} \quad (\text{B.9})$$

$$h^{\mu\nu} \equiv \eta^{\underline{\mu}\underline{\alpha}} \delta_{\alpha}^{\nu} f_{\underline{\mu}}^{\mu} = \eta^{\nu\alpha} \delta_{\underline{\alpha}}^{\mu} f_{\underline{\mu}}^{\nu} \quad (\text{B.10})$$

so that we have the linear relations

$$g_{\mu\nu} \equiv e_{\mu}^{\underline{\mu}} e_{\nu}^{\underline{\nu}} \eta_{\underline{\mu}\underline{\nu}} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2) \quad (\text{B.11})$$

$$g^{\mu\nu} \equiv e_{\underline{\mu}}^{\mu} e_{\underline{\nu}}^{\nu} \eta^{\underline{\mu}\underline{\nu}} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2) \quad (\text{B.12})$$

$$g_{\mu\nu} g^{\nu\alpha} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\alpha} - h^{\nu\alpha}) + \mathcal{O}(h^2) = \delta_{\mu}^{\alpha} + \mathcal{O}(h^2) \quad . \quad (\text{B.13})$$

Enforcing the supersymmetry on the Lagrangian (B.1) such that

$$D_a \mathcal{L} = 0 + \text{total derivatives} \quad (\text{B.14})$$

leads to the following solution for the f_i in terms of $s_1, p_1, a_0, s_0, p_0, h_0, f_0, a_1, a_2, h_1$, and h_2

$$\begin{aligned} f_1 &= -\frac{s_0}{2f_0} s_1 \quad , \quad f_2 = -\frac{p_0}{2f_0} p_1 \quad , \quad f_4 = -\frac{a_0}{2f_0} a_1 \quad , \\ f_3 &= -\frac{a_0}{f_0} \left(a_2 - \frac{1}{2} a_1 \right) \quad , \quad f_5 = -2\frac{h_0}{f_0} h_1 \quad , \quad h_2 = f_6 = f_8 = f_9 = 0 \quad . \end{aligned} \quad (\text{B.15})$$

The fact that f_7 is yet unconstrained is not surprising as this term encodes the gauge symmetry in Eq. (2.8). For now, we shall keep it unknown.

This gives us, for transformation laws that are a symmetry of the Lagrangian (B.1)

$$D_a S = i s_1 (\sigma^{\mu\nu})_a^b \partial_{\mu} \psi_{\nu b} \quad (\text{B.16a})$$

$$D_a P = p_1 (\gamma^5 \sigma^{\mu\nu})_a^b \partial_{\mu} \psi_{\nu b} \quad (\text{B.16b})$$

$$D_a A_\mu = i a_1 (\gamma^5 \gamma^\nu)_a{}^b \partial_{[\nu} \psi_{\mu]b} + a_2 \epsilon_\mu{}^{\nu\alpha\beta} (\gamma_\nu)_a{}^b \partial_\alpha \psi_{\beta b} \quad (\text{B.16c})$$

$$D_a h_{\mu\nu} = h_1 (\gamma_\mu)_a{}^b \psi_{\nu b} + h_1 (\gamma_\nu)_a{}^b \psi_{\mu b} \quad (\text{B.16d})$$

$$\begin{aligned} D_a \psi_{\mu b} = & -i \frac{s_0}{2f_0} s_1 (\gamma_\mu)_{ab} S - \frac{p_0}{2f_0} p_1 (\gamma^5 \gamma_\mu)_{ab} P - \frac{a_0}{f_0} \left(a_2 - \frac{1}{2} a_1 \right) (\gamma^5)_{ab} A_\mu + \\ & -i \frac{a_0}{2f_0} a_1 (\gamma^5 \sigma_\mu{}^\nu)_{ab} A_\nu - 2 \frac{h_0}{f_0} h_1 (\sigma^{\alpha\beta})_{ab} \partial_\alpha h_{\beta\mu} + i f_7 C_{ab} \partial_\mu h \end{aligned} \quad (\text{B.16e})$$

B.2 Algebra

Next, we wish to finish solving for the leftover constants in the transformation laws (B.16) so they satisfy the algebra:

$$\begin{aligned} \{D_a, D_b\} S &= 2i (\gamma^\mu)_{ab} \partial_\mu S \quad , \quad \{D_a, D_b\} P = 2i (\gamma^\mu)_{ab} \partial_\mu P, \\ \{D_a, D_b\} A_\nu &= 2i (\gamma^\mu)_{ab} \partial_\mu A_\nu \quad , \end{aligned} \quad (\text{B.17})$$

$$\{D_a, D_b\} h_{\mu\nu} = 2i (\gamma^\alpha)_{ab} \partial_\alpha h_{\mu\nu} + 2i \partial_{(\mu} (V_{\nu)})_{ab}, \quad (\text{B.18})$$

$$\{D_a, D_b\} \psi_{\mu c} = 2i (\gamma^\alpha)_{ab} \partial_\alpha \psi_{\mu c} + 2i \partial_\mu \varphi_{abc} \quad . \quad (\text{B.19})$$

with the gauge freedom encoded by

$$(V_\nu)_{ab} = b_1 (\gamma^\alpha)_{ab} h_{\nu\alpha} + b_2 (\gamma_\nu)_{ab} h \quad (\text{B.20})$$

$$\varphi_{abc} = c_1 (\gamma^\alpha)_{ab} \psi_{\alpha c} + c_2 (\gamma^\alpha)_{c(a} \psi_{|a|b)} + [c_3 (\gamma^5 \gamma^\sigma)_{c(a} (\gamma^5)_b^d + c_4 C_{c(a} (\gamma^\sigma)_b^d] \psi_{\sigma d} \quad (\text{B.21})$$

with b_i and c_i new constants to be solved for. We in fact started from a much more complicated gauge invariant term than φ_{abc} , but here only summarize the part of the proof for the terms that did not vanish.

By direct, brute force calculation, we have found that the most general set of parameters which satisfy closure as above and Lagrangian invariance are

$$\begin{aligned} s_0 &= \frac{2f_0}{3s_1^2} \quad , \quad p_0 = \frac{2f_0}{3p_1^2} \quad , \quad a_0 = \frac{2f_0}{3a_1^2} \quad , \quad h_0 = \frac{f_0}{4h_1^2} \quad , \\ a_2 &= -\frac{1}{2} a_1 \quad , \quad f_1 = -\frac{1}{3s_1} \quad , \quad f_2 = -\frac{1}{3p_1} \quad , \quad f_3 = \frac{2}{3a_1} \quad , \\ f_4 &= -\frac{1}{3a_1} \quad , \quad f_5 = -\frac{1}{2h_1} \quad , \quad b_1 = -\frac{1}{2} \quad , \quad c_1 = -1 \quad , \\ c_2 &= c_3 = c_4 = \frac{1}{4} \quad , \quad f_7 = b_2 = f_6 = h_2 = f_8 = f_9 = 0 \quad . \end{aligned} \quad (\text{B.22})$$

There are still has five free parameters, s_1, p_1, a_1, h_1 , and f_0 , which encode the left over normalizations in the Lagrangian, an overall rescaling of the Lagrangian, and

the binary symmetries of the Lagrangian, i.e. $h_{\mu\nu} \rightarrow -h_{\mu\nu}$, etc. Notice, that closure has now forced $h_7 = b_2 = 0$.

The final, most general Lagrangian, transformation laws, and algebra are:

$$\begin{aligned} \mathcal{L} = & \frac{f_0}{4h_1^2} \left(-\frac{1}{2} \partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} + \frac{1}{2} \partial^\alpha h \partial_\alpha h - \partial^\alpha h \partial^\beta h_{\alpha\beta} + \partial^\mu h_{\mu\nu} \partial_\alpha h^{\alpha\nu} \right) + \\ & - \frac{2f_0}{3s_1^2} \frac{1}{2} S^2 - \frac{2f_0}{3p_1^2} \frac{1}{2} P^2 + \frac{2f_0}{3a_1^2} \frac{1}{2} A_\mu A^\mu - f_0 \frac{1}{2} \psi_{\mu a} \epsilon^{\mu\nu\alpha\beta} (\gamma^5 \gamma_\nu)^{ab} \partial_\alpha \psi_{\beta b} \end{aligned} \quad (\text{B.23})$$

and

$$D_a S = i s_1 (\sigma^{\mu\nu})_a{}^b \partial_\mu \psi_{\nu b} \quad (\text{B.24a})$$

$$D_a P = p_1 (\gamma^5 \sigma^{\mu\nu})_a{}^b \partial_\mu \psi_{\nu b} \quad (\text{B.24b})$$

$$D_a A_\mu = i a_1 (\gamma^5 \gamma^\nu)_a{}^b \partial_{[\nu} \psi_{\mu] b} - \frac{1}{2} a_1 \epsilon_\mu{}^{\nu\alpha\beta} (\gamma_\nu)_a{}^b \partial_\alpha \psi_{\beta b} \quad (\text{B.24c})$$

$$D_a h_{\mu\nu} = h_1 (\gamma_\mu)_a{}^b \psi_{\nu b} + h_1 (\gamma_\nu)_a{}^b \psi_{\mu b} \quad (\text{B.24d})$$

$$\begin{aligned} D_a \psi_{\mu b} = & -\frac{i}{3s_1} (\gamma_\mu)_{ab} S - \frac{1}{3p_1} (\gamma^5 \gamma_\mu)_{ab} P + \frac{2}{3a_1} (\gamma^5)_{ab} A_\mu - \frac{i}{3a_1} (\gamma^5 \sigma_\mu{}^\nu)_{ab} A_\nu + \\ & - \frac{1}{2h_1} (\sigma^{\alpha\beta})_{ab} \partial_\alpha h_{\beta\mu} \end{aligned} \quad (\text{B.24e})$$

which satisfy the algebra in Eq. (2.5). With no loss of generality, we therefore make the following choices for the final five parameters

$$a_1 = s_1 = p_1 = f_0 = 2h_1 = 1 \quad (\text{B.25})$$

which puts the Lagrangian and transformation laws into the forms used throughout the paper, i.e., Eqs. (2.2) and (2.1). Also, various gamma matrix identities take us from Eqs. (B.21) and (B.22) to Eq. (2.6) for the closure term on the gravitino.

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